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Topology Optimization of Two Linear Elastic Bodies in Unilateral Contact

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Abstract
The optimal solutions are most sensitive to the boundary conditions when performing topology optimization of components. In many applications the design domain of the components are subjected to unilateral contact conditions. In order to obtain relevant conceptual designs by topology optimization of such systems, the contact conditions should be included explicitly in the optimization. Recently, in a number of works by Strömberg and Klarbring, such a method has been developed for one elastic body unilateral constrained to rigid supports. Here, this approach is extended such that a system of two elastic bodies in unilateral contact is considered. For this systems the compliance is minimized by adopting the SIMP-model. A nested formulation of the problem is solved by SLP, where the sensitivities are obtained by solving an adjoint equation. In this latter equation, the Jacobian from the Newton method used to solve the state problem appears. The state problem is treated by an augmented Lagrangian formulation of the bodies in contact. Thus, the Jacobian is simply the gradient of the corresponding system of equations to this formulation. The method is implemented in the toolbox Topo4abq by using Matlab and Intel Fortran. The method is both efficient and robust. This is demonstrated by solving several 2D-problems. The results are also compared to the solutions obtained when the contact conditions are treated by joining the two bodies to one body. In a near future 3D-problems will also be solved by using the presented approach.
Keywords: Topology optimization, SIMP, Contact, SLP.

1. Introduction

Most recently, topology optimization of linear elastic structures unilateral constrained to rigid foundations have been investigated by the author and Klarbring in a number of papers [1, 2, 3]. The SLP approach developed in these works seems most promising. In this paper, this approach is investigated for two linear elastic bodies in contact.

The state problem is formulated by using an augmented Lagrangian formulation. This is done for a node-to-node formulation of the contact interface. The corresponding optimality conditions are then solved by a Newton method with an inexact line-search procedure. This approach produces most accurate state solutions which has been demonstrated in several works. For instance, in the original implementation by Strömberg [4], wear problems were solved successfully. In Strömberg [5], the solutions to three-dimensional friction problems were found using this approach. Furthermore, thermomechanical problems were solved in [6, 7]. The dynamic transmission error was studied in [8]. Recently, non-local damage coupled to wear was solved by using this Newton approach in [9].

The optimization problem is solved by sequential linear programming (SLP). This is done by a nested approach, where the state is considered to be a function of the design parameters by solving the state equations implicitly. The design parametrization is realized by adopting the SIMP-model. The sensitivity analysis is then performed by defining an adjoint equation. This equation is defined by use of the Jacobian appearing in the Newton method when the search direction is determined. After the sensitivities have been filtered by Sigmund's filter, the final LP-problem is solved by the interior point method that is available in the optimization toolbox of Matlab. Concerning details about SLP, SIMP, the nested approach, the adjoint equation and filters, see e.g. the textbooks by Bendsøe and Sigmund [10], and Christensen and Klarbring [11], respectively.

Two different contact problems are optimized for different settings. The solutions are compared to the solutions obtained if the contact boundaries are merged together. In such manner it is demonstrated that it might sometimes be necessary in some applications to include the contact conditions when assemblies of design domains are optimized.

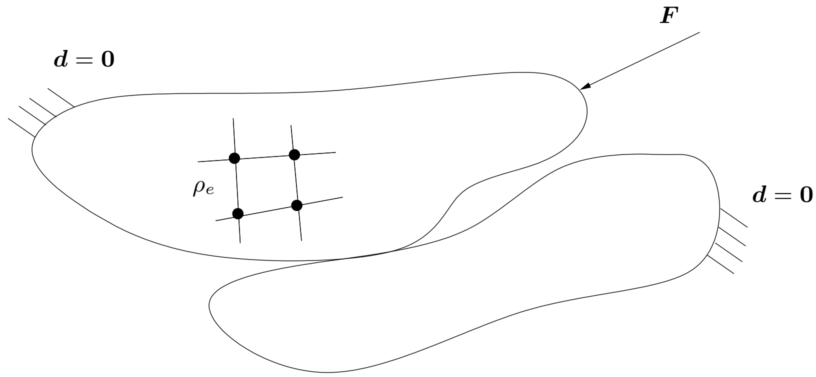

Figure 1: *A system of two linear elastic bodies in unilateral contact.*

2. The State Problem

Let us consider two linear elastic bodies parameterized by adopting the SIMP-modell. That is, the global stiffness matrix is generated by the following assembly procedure

$$\boldsymbol{K}=\boldsymbol{K}(\boldsymbol{\rho})=\bigcup_{e=1}^{n_{\mathrm{el}}}\rho_e^n\boldsymbol{k}_e, \tag{1}$$

where $\boldsymbol{\rho}=\{0<\rho_e\leq 1\}$, $n\geq 1$, $\boldsymbol{k}_e$ is a local element stiffness matrix and n_{el} is the total number of finite elements.

Contact might be developed between the bodies. This is treated with a node-to-node contact formulation. The contact nodes belonging to one of the bodies are denoted slave nodes and the remaining set of contact nodes are called master nodes. Outward unit normals $\boldsymbol{n}$ for both the slave nodes and the master nodes are identified. By using these normals, we define the following transformation matrices:

$$\boldsymbol{C}_s=[\boldsymbol{C}_s^{\mathrm{row}}=\boldsymbol{n}],\qquad \boldsymbol{C}_m=[\boldsymbol{C}_m^{\mathrm{row}}=\boldsymbol{n}], \tag{2}$$

such that the normal displacements of the slave and master nodes are obtained by

$$\boldsymbol{d}_s=\boldsymbol{C}_s\boldsymbol{d},\qquad \boldsymbol{d}_m=\boldsymbol{C}_m\boldsymbol{d}, \tag{3}$$

where $\boldsymbol{d}$ is the nodal displacement vector.

For a given external force $\boldsymbol{F}$, equilibrium for the two bodies reads

$$\boldsymbol{K}(\boldsymbol{\rho})\boldsymbol{d}+(\boldsymbol{C}_s+\boldsymbol{C}_m)\,\boldsymbol{P}_N=\boldsymbol{F}. \tag{4}$$

In this expression, the contact forces $\boldsymbol{P}_N$ are defined by

$$\boldsymbol{P}_N=\left(\boldsymbol{P}_N+r(\boldsymbol{d}_s+\boldsymbol{d}_m-\boldsymbol{g})\right)_+. \tag{5}$$

where $r>0$ is a given penalty coefficient, $\boldsymbol{g}$ contains the initial gaps between the slave and master nodes, and

$$(x)_+=\frac{x+|x|}{2}. \tag{6}$$

The projection in (5) is equivalent to Signorini's contact conditions and constitutes the basis in an augmented Lagrangian formulation, see e.g. Strömberg [4].

For a given density distribution $\boldsymbol{\rho}=\hat{\boldsymbol{\rho}}$, the state $\boldsymbol{x}=\{\boldsymbol{d};\boldsymbol{P}_N\}$ is found by solving (4) and (5) simultaneously by using Newton's method with an Armijo line-search procedure. Letting $\boldsymbol{h}=\boldsymbol{h}(\boldsymbol{\rho},\boldsymbol{x})$ represent (4) and (5), the search direction at differentiable state is

$$\boldsymbol{s}=-\left(\nabla_x\boldsymbol{h}(\hat{\boldsymbol{\rho}},\boldsymbol{x})\right)^{-1}\boldsymbol{h}(\hat{\boldsymbol{\rho}},\boldsymbol{x}). \tag{7}$$

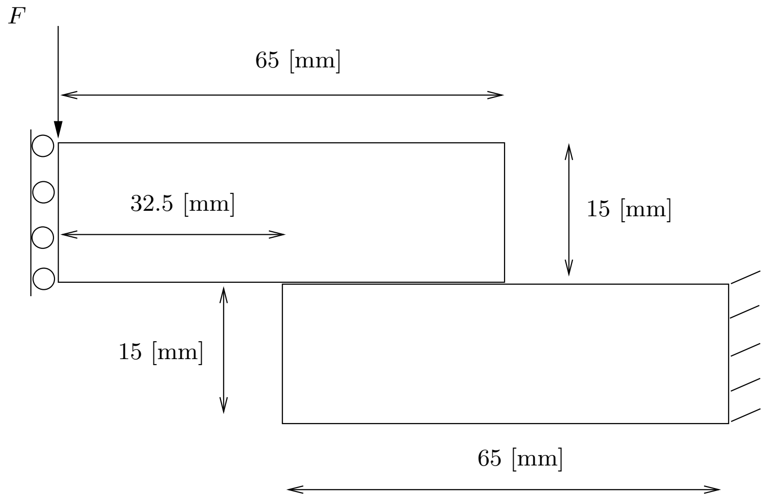

Figure 2: *The geometry, the boundary conditions and the external force $F = 1\text{E}4$ [N] for the first problem.*

The state is then generated by the following sequence:

$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^n + \alpha \boldsymbol{s}, \tag{8}$$

where α is the line-search parameter. Details about the treatment at non-differentiable states are presented in [4].

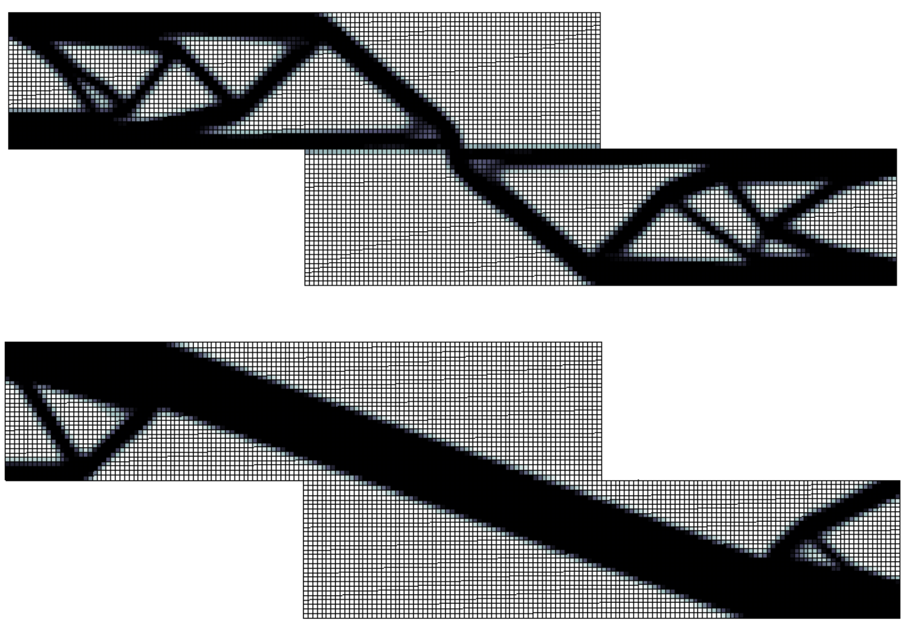

Figure 3: *The solution with and without contact conditions, respectively.*

2. The Optimization Problem

For the state problem presented above, the compliance

$$c = \boldsymbol{F}^T \boldsymbol{d} \tag{9}$$

is minimized for a given constraint on the volume

$$V = V(\boldsymbol{\rho}) = \sum_{e=1}^{n_{\mathrm{el}}} \rho_e V_e, \tag{10}$$

where V_e is the local element volume. Thus, the following optimization problem is stated

$$\begin{cases} \min\limits_{(\boldsymbol{\rho},\boldsymbol{x})} c = \boldsymbol{F}^T \boldsymbol{d} \\ \text{s.t.} \begin{cases} \boldsymbol{h}(\boldsymbol{\rho},\boldsymbol{x}) = \boldsymbol{0} \\ V(\boldsymbol{\rho}) \leq \hat{V} \\ 0 < \rho_e \leq 1. \end{cases} \end{cases} \tag{11}$$

This is solved by first rewriting the problem using a nested approach and then applying sequential linear programming. The nested problem, the sensitivity analysis and the corresponding LP-problem are presented below in detail.
We consider $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{\rho})$ to be explicitly given by solving $\boldsymbol{h}(\boldsymbol{\rho},\boldsymbol{x}) = \boldsymbol{0}$. The nested problem then reads

$$\begin{cases} \min\limits_{\boldsymbol{\rho}} c(\boldsymbol{\rho}) = \boldsymbol{F}^T \boldsymbol{d}(\boldsymbol{\rho}) \\ \text{s.t.} \begin{cases} V(\boldsymbol{\rho}) \leq \hat{V} \\ 0 < \rho_e \leq 1. \end{cases} \end{cases} \tag{12}$$

The sensitivities

$$\xi_e = \frac{\partial c}{\partial \rho_e} = \begin{Bmatrix} \boldsymbol{F} \\ \boldsymbol{0} \end{Bmatrix}^T \frac{\partial \boldsymbol{x}}{\partial \rho_e} \tag{13}$$

are determined by introducing an adjoint equation, i.e.

$$(\nabla_x \boldsymbol{h})^T \boldsymbol{\gamma} = \begin{Bmatrix} \boldsymbol{F} \\ \boldsymbol{0} \end{Bmatrix}. \tag{14}$$

Putting this in (13) yields

$$\xi_e = \boldsymbol{\gamma}^T \nabla_x \boldsymbol{h} \frac{\partial \boldsymbol{x}}{\partial \rho_e}. \tag{15}$$

Furthermore, taking the partial derivative of $\boldsymbol{h} = \boldsymbol{0}$ with respect to ρ_e implies

$$\frac{\partial \boldsymbol{h}}{\partial \rho_e} + \nabla_x \boldsymbol{h} \frac{\partial \boldsymbol{x}}{\partial \rho_e} = \boldsymbol{0}. \tag{16}$$

(16) inserted in (15) results in

$$\xi_e = -\boldsymbol{\gamma}^T \frac{\partial \boldsymbol{h}}{\partial \rho_e}. \tag{17}$$

Before formulating the corresponding LP-problem by using the sensitivities above, ξ_e are filtered using Sigmund's filter. The filtered sensitivities are denoted ξ_e^s.
Finally, we solve the following LP-problem in sequence:

$$\begin{cases} \min\limits_{\boldsymbol{\rho}} \sum_{e=1}^{n_{\mathrm{el}}} \xi_e^s \rho_e \\ \text{s.t.} \begin{cases} \sum_{e=1}^{n_{\mathrm{el}}} V_e \rho_e - \hat{V} \leq 0 \\ \rho_e^l \leq \rho_e \leq \rho_e^u \end{cases} \end{cases} \tag{18}$$

by using the interior point method of Matlab. ρ_e^l and ρ_e^u appearing in (18) represent the move limits.

3. Numerical Examples

The method presented above is implemented in Topo4abq. Topo4abq (*www.fema.se*) is a toolbox for topology optimization developed by using Matlab and Intel Fortran. The toolbox runs on both 32-bits and 64-bits versions of Windows. The toolbox has a user-friendly GUI and it is compatible with the Abaqus/CAE environment which makes it easy to use by students and engineers.

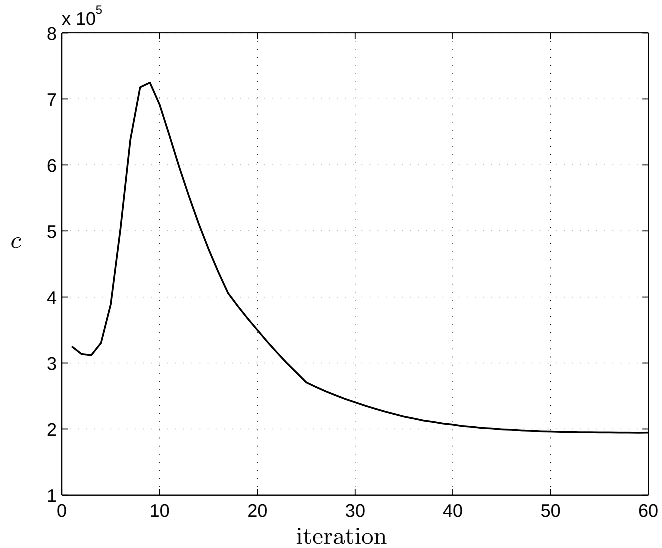

Figure 4: *Convergence history for the contact problem.*

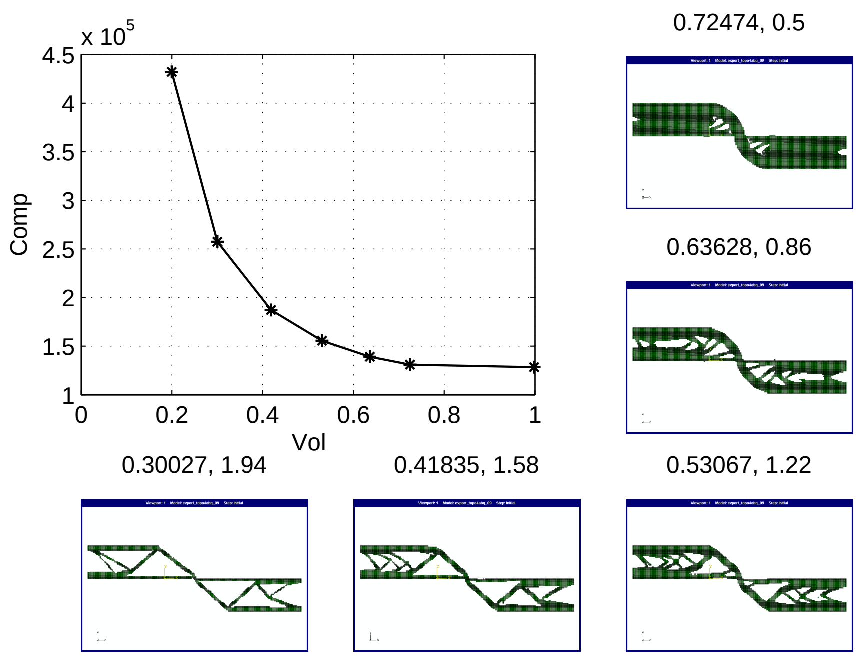

Figure 5: *Tradeoff between the compliance and the volume fraction for the contact problem. The design parameter* η *is taken to be in the range of [0.5:0.36:2.3].*

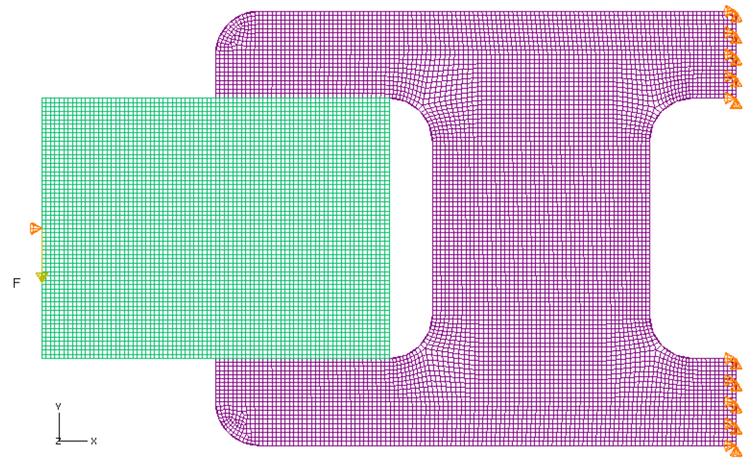

Figure 6: *The meshes, the boundary conditions and the force for the second problem.*

The solutions of two different kind of problems for different settings are presented in this section. Both problems are two-dimensional. The method is also implemented for three-dimensional geometries. The solutions of 3D-problems will be presented at the conference.

Firstly, a contact problem of three bodies are considered. However, the symmetry is utilized such that only two bodies are solved numerically by the proposed method. The geometry, boundary conditions and the load of the problem are presented in Figure 2. 7880 fully integrated 4-noded isoparametric bilinear elements represent the bodies. The plain strain is utilized, Young's modulus is 2.1E5 [Pa] and Poisson's ratio is 0.3. The filter radius is 0.9 [mm] and the limit on the volume fraction is 0.4. The solution is presented at the top of Figure 3. Below, at the bottom of Figure 3, the solution obtained when the contact boundaries are merged together is presented. It is clear that the two solutions are very different. The convergence history of the contact solution is presented in Figure 4. During the first ten iterations the SIMP factor n is increased from 1 to 3 by a log-sigmoid relationship. This explains the peak in the beginning of the history.

In Figure 5 the tradeoff between the compliance and the volume fraction is also plotted for the contact problem. The tradeoff curve is generated by minimizing the following objective:

$$f = cV^{\eta} \tag{19}$$

for different values on the design parameter η. This objective was suggested in Strömberg [3], which is a generalization of the hyperoptimal formulation suggested by Rozvany et al. [12]. The picture in Figure 5 is automatically generated as an output from Topo4abq.

The second problem is a contact problem between two bodies with two separated contact interfaces. The meshes, boundary conditions and the applied force are presented in Figure 6. 35560 elements with the same properties as the elements of the previous problem are used for the meshes of the bodies. Solutions for three different settings of this problem are presented in Figure 7. First, at the top of this figure, the solution for the basic settings discussed above is plotted. Secondly, in the middle of the figure, an additional load case is added. An opposite force of equal size is also considered. For a linear problem, these two load cases would produce the same solution. However, this is not the case for contact problems. It is obvious that these two solutions differ a lot. Finally, at the bottom of the figure, the solution obtained when the contact boundaries are merged together is presented. Apparently, this solution is very different from the other two solutions.

4. Summary

In this work, a method for topology optimization of a linear elastic structure unilaterally constrained to rigid supports is extended such that systems of several elastic bodies in contact are optimized. The method is both efficient and robust, which is demonstrated by solving two-dimensional problems for different settings. The importance of including the contact conditions is also demonstrated by comparing

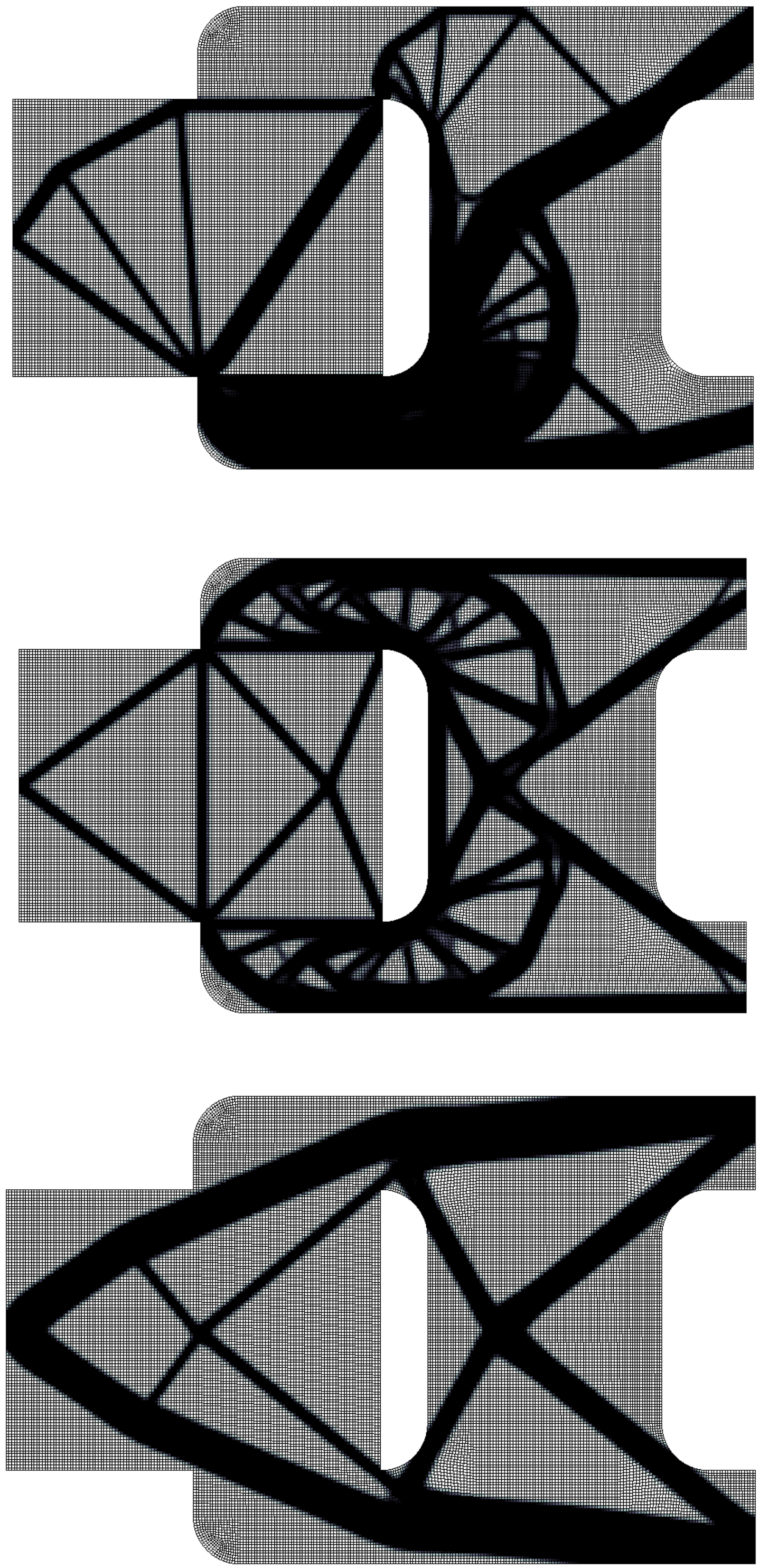

Figure 7: *Solutions for three different settings of the second problem. At the top of the figure the solution for the basic settings is presented, in the middle two load cases are considered and finally, at the bottom, the contact boundaries are merged together.*

the solution to the solutions obtained when the contact boundaries are merged together. The optimal solutions are very different.

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