Abstract
A kind of nonconforming finite element with the potential to reduce computational cost of the whole topology optimization process is introduced. The reduction is obtained due to its linear shape functions which can be defined based on its vertices. This element is called the P1-nonconforming finite element (P1NC). Topology optimization problems involving incompressible materials are solved to demonstrate the unique characteristics and applicability of P1NC. Locking-free property inherent to nonconforming finite elements enables the use of P1NC with pure displacement formulation free from the numerical instability associated with material incompressibility. Mean compliance minimization problems for structures with incompressible materials and channel design problems for Stokes flow are solved based on finite element analysis by using the proposed P1NC. Implementation of the finite element analysis using this element is straightforward since it is not much different from that for bilinear conforming elements. Results show that P1NC can efficiently reduce numerical cost for topology optimization method involving incompressible materials especially in large-sized problems such as three-dimensional problems.

Keywords: Topology, P1-nonconforming, locking-free, incompressible, Stokes

1. Introduction
Topology optimization is a numerical method for finding the best material configuration that meets and best performs the required structural function. Since its introduction, it has been implemented to a multitude of application for academic, commercial and industrial purposes. Among many applications, problems involving material incompressibility (e.g. for rubber and fluid) pose challenging issues. The major issue is brought by the locking effect (Poisson’s locking or volumetric locking) wherein the material approaching incompressibility ($\nu \rightarrow 0.5$) restricts itself against deformations. This behavior results to a too stiff structural response. In effect, prohibits the use of displacement method-based finite elements which are commonly used in classical topology optimization problems.

Mixed methods are among the suggested solutions to locking effect. The work by Sigmund and Clausen [1] implemented a variant of such methods to topology optimization of pressure-loaded structures. The main idea was to represent a fixed void region as hydrostatic fluid so that through this region pressure can be imposed and transferred to the design domain. Hydrostatic fluid property was defined through the bulk and shear modulus. Shortly after the success of [1], few other methods offering solutions to topology optimization problem involving incompressibility have emerged. Bruggi and Veninni [2] utilized similar material interpolation scheme with [1] and work entirely on topology optimization of incompressible materials using the mixed formulation with composite finite element suggested by Johnson and Mercier [3]. It can be noted that the core of the approaches cited earlier lies on the finite element formulation i.e. mixed formulation.

Alternatively, problems involving incompressibility can be solved using the nonconforming finite elements [4; 5; 6; 7; 8; 9]. These nonconforming finite elements are characterized by nodes located only at the midpoints of edges (for 2D) and centers of faces (for 3D). The node configuration enables the element to deform freely overcoming the locking effect. Hence, Jang and Kim [10] introduced a pure displacement formulation using the so called DSSY-nonconforming element proposed in [7] and solved topology optimization problems involving incompressible materials. The same finite element formulation was further implemented to the optimal design of compliant mechanism actuated by pressure [11]. This method however, uses nonconforming elements having basis functions with high-order polynomials (at least quadratic). This can be disadvantageous for large size problems as in 3D.

In this article, we present the P1-nonconforming element (P1NC) developed by Park and Sheen [12] that can be used not only to solve problems involving incompressibility but significantly reduced computation cost. Just like other nonconforming elements, the locking-free property of P1NC waives the issue of incompressibility. This
element has linear basis functions and its degrees of freedom (dof) are defined on the element vertices. Fewer dof is needed to define a high resolution topology compared to any other methods mentioned above. For instance in an \( N \times N \) mesh, the dof are \( 7N^2 + 10N + 3, 4N^2 + 4N \), and \( 2N^2 + 4N + 2 \), for mixed displacement-pressure method, high-order DSSY nonconforming element, and P1-nonconforming element, respectively. Furthermore, the dof defined on the vertices enables the easy implementation of P1NC to the standard conforming finite element code. It only requires a slight modification of the basis functions. Few mean compliance minimization problems for structures with incompressible materials and channel design problems for Stokes flow are solved to validate the effectiveness of presented approach.

The thrust of the presented approach is the potential reduction of computation cost of the whole topology optimization procedure due to the remarkable properties of P1-nonconforming element. Aside from 3D problems, this effect enables the use of finer mesh to 2D problems requiring high clarity images as in fluidic membrane.

2. P1-nonconforming element (P1NC)

The basis functions of P1NC (see Figure 1) derived from the space of linear polynomials \( P_1(Q) = \text{span}\{1, x, y\} \) defined in quadrilateral element, \( Q \) are given as

\[
\phi_i(m_k) = \begin{cases} 
1, & k = j, j + 1 \text{ mod } 4, \\
0, & \text{otherwise}
\end{cases}
\] (1)

where \( m_k \) (\( k = 1, 2, 3, 4 \) and \( m_k = m_j \)) denotes the midpoint of the \( k \)th edge of the element. It can be shown that any three of \( \{\phi_1, \phi_2, \phi_3, \phi_4\} \) can span the space \( P_1(Q) \). Moreover, Eq.(1) implies that on an element \( Q \), a function \( u \in P_1(Q) \) should satisfy

\[
u(m_j) + u(m_i) = \frac{u(v_j) + u(v_i)}{2} + \frac{u(v_2) + u(v_j)}{2} = \frac{u(v_1) + u(v_i)}{2} + \frac{u(v_j) + u(v_i)}{2} = u(m_2) + u(m_j),
\] (2)

where \( v_j \) (\( j = 1, 2, 3, 4 \)) are vertices of \( Q \). Hence, the dimension of the P1NC space, \( \mathcal{N}_h \), can be written as

\[
\dim(\mathcal{N}_h) \leq N_e - N_v = N_v - 1
\] (3)

where \( N_o, N_e \) and \( N_v \) are the numbers of elements, element edges and vertices, respectively (see [12] for details).

![Figure 1: P1-nonconforming element](image)

For a rectangular element, it is easily seen that \( \phi_i = 3/2 \) on the vertex \( v_j \). Thus basis function defined on the vertices can be explicitly expressed as

\[
\phi_i = \frac{1}{2}(1 - \xi - \eta)
\] (4a)

\[
\phi_j = \frac{1}{2}(1 + \xi - \eta)
\] (4b)
where \((\xi, \eta)\) is the element coordinate defined in \([-1,1]^2\). Figure 2 shows the location of vertex \(v_j\) in a rectangular mesh and the shape functions associated with it as defined in Eq.(4).

\[
\varphi_j = \frac{1}{2}(1 + \xi + \eta) \quad (4c)
\]
\[
\varphi_i = \frac{1}{2}(1 - \xi + \eta) \quad (4d)
\]

Figure 2: Rectangular finite element mesh that defines \(v_j\) (left) and the associated P1NC shape function (right)

The convergence of the P1NC is expressed similar to that of the high-order DSSY nonconforming element (see [7] and [12] for mathematical derivation). In Figure 3, convergences of the conforming bilinear element, the DSSY-nonconforming element and the P1NC are compared for a simple cantilever beam problem. It can be observed that the convergence rate of P1NC is comparable to the high-order nonconforming and the bilinear conforming counterparts.

The P1-nonconforming element is free from the Poisson locking effect of an incompressible material. In the case of DSSY nonconforming element proof of such property had been shown in [10]. Since the same concept applies to P1NC, the reader is referred to the cited work for details.

Figure 3: Solution convergence of P1NC for a cantilever problem

3. Verification and discussions

3.1 Topology optimization involving incompressible material

The typical mean compliance minimization problem is written as

Minimize: \( c(\rho) = f^T u \), \quad (5a)\\

Subject to: \( \sum_{i=1}^{n} \rho_i v_i \leq V \), \quad (5b)\\

\[ \mathbf{K}u = f \] \quad (5c)
where \( \mathbf{K}, \mathbf{u} \) and \( \mathbf{f} \) are the stiffness matrix, displacement vector and the force vector, respectively, \( \nu_e \) is the element volume, \( V' \) is the predefined volume limit and \( N_p \) is the number of design variables. We note that Eq.(5c) is a pure-displacement formulation of the finite element analysis with P1NC. Adopted from [1], the material interpolation for an element \( e \) corresponding to the design \( \rho_e \) is expressed as

\[
K(\rho_e) = K_e = K_{\text{solid}} + \rho_e^p (K_{\text{solid}} - K_{\text{void}}),
\]

and

\[
G(\rho_e) = G_e = G_{\text{solid}} + \rho_e^p (G_{\text{solid}} - G_{\text{void}}),
\]

where \( p \) is the penalty. This way the material incompressibility is easily imposed by assigning suitable values to the bulk modulus \( K \) and shear modulus \( G \). Eq.(6) comes into play in the element stiffness matrix

\[
\mathbf{K} = \sum_{e=1}^{N_e} \int_{\Omega_e} \mathbf{B}^T \mathbf{D} \mathbf{B} d\Omega
\]

via the elasticity tensor

\[
\mathbf{D} = \begin{bmatrix}
K_e + G_e & K_e - G_e & 0 \\
0 & K_e + G_e & 0 \\
0 & 0 & G_e
\end{bmatrix}.
\]

The example in Figure 4 deals with optimal design of rubber mount that can be used to isolate vibration in mechanical structures. The design domain is shown in Figure 4(left). By symmetry, only half is discretized with 40 \( \times \) 80 P1NC. The final design is expected to be consists of rubber material with cavities for fluid inclusions. Thus material interpolation is defined such that solid regions represent rubber and void represent solid i.e. \( K_{\text{solid}} = K_{\text{solid/flu}} = 100 \) and shear modulus, \( G_{\text{solid}} = 0.3333 \) and \( G_{\text{solid/flu}} = 0.001 \). The penalty exponent \( p = 5 \). The solid region is constrained to 65% of the total volume including the fixed solid region. The optimized design is shown in Figure 4(right), which is very similar to that obtained by DSSY nonconforming [10].

![Figure 4: (a) Design domain and (b) optimization result of the simplified rubber mount problem](image)

3.2 Channel design problems for fluids in Stokes flow

The application of P1NC is extended to fluid mechanics by designing channels with fluid defined by the Stokes equations. Single field (velocity field) formulation will be developed using the Galerkin approach. The governing Stokes equations for an incompressible Newtonian fluid with constant viscosity are

\[
-\nabla \cdot (\mu \nabla \mathbf{v}) + \nabla p = \mathbf{f}, \quad \nabla \cdot \mathbf{v} = 0,
\]

where \( p \) is the pressure, \( \mathbf{v} = \{v_x, v_y, v_z\}^T \) is the fluid velocity, \( \mathbf{f} = \{f_x, f_y, f_z\}^T \) is the body force, and \( \mu \) is the viscosity. The pressure can be expressed as

\[
p = -\kappa \dot{\varepsilon}_v = -\kappa \nabla \cdot \mathbf{v},
\]

where \( \kappa \) is the volumetric viscosity coefficient analogous to the bulk modulus \( K \) in solid mechanics and \( \dot{\varepsilon}_v \) is the volumetric strain rate. Incompressibility can be imposed by setting very large value to \( \kappa \).
Figure 5: The panel flow model defined by the 3D domain, \( V = \Omega \times [-h/2, h/2] \) and the analysis domain \( \Omega = x \times y \).

Considering the panel flow model illustrated in Figure 5 and by substituting Eq.(9) to Eq.(8) the Stokes equation of the panel flow for topology optimization based from [13], is written as

\[
-\nabla \cdot (\mu \nabla \tilde{v}) - \nabla \cdot (\kappa \nabla \cdot \tilde{v}) + \alpha \tilde{v} = \tilde{f}
\]

(11)

where \( \tilde{v} = \{\tilde{v}_x, \tilde{v}_y\}^T \) is a two-dimensional velocity defined on \( \Omega \) and \( \alpha \) is inverse permeability. Consequently, the three-dimensional velocity \( \mathbf{v}(x, y, z) = [1 - (2/\alpha)^2] \tilde{v}(x, y) \).

The derived weak form of Eq.(11) is

\[
\int_{\partial \Omega} (\mu \nabla \tilde{v} \cdot \mathbf{n} \tilde{w} + \kappa \nabla \cdot \tilde{v} \cdot \mathbf{n} \tilde{w} + \alpha \tilde{v} \cdot \mathbf{n} \tilde{w}) \, ds - \int_{\partial \Omega} \tilde{f} \cdot \tilde{w} \, dA
\]

(12)

where \( \mathbf{n} \) is the outward unit surface normal to \( \partial \Omega \). After discretization, Eq.(13) we obtain the system equilibrium equation similar to Eq.(5c) except that this time the field variable is the fluid velocity. The element stiffness

\[
\mathbf{k}_e = \mathbf{k}_{v,v} + \mathbf{k}_{v,w} = \int_{\Omega} [\mu \mathbf{C} + \kappa \mathbf{E}^T \mathbf{E} + \alpha \mathbf{N}^T \mathbf{N}] \, d\Omega
\]

(13)

with \( \mathbf{C} = \mathbf{L} \mathbf{N} \), \( \mathbf{E} = \mathbf{M} \mathbf{N} \), \( \mathbf{L} \) and \( \mathbf{M} \) are differential operator matrices expressed as

\[
\mathbf{L} = \begin{bmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 & 0 \\
0 & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y}
\end{bmatrix}^T, \quad \mathbf{M} = \begin{bmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y}
\end{bmatrix}.
\]

(14)

The material interpolation is characterized by the parameter \( \alpha \) in Eq.(13) written as [13]

\[
\alpha = \bar{\alpha} + (\alpha - \bar{\alpha}) \rho \frac{1+q}{\rho + q}.
\]

(15)

The optimization goal is to find the optimum channel topology that minimizes the total potential power. The problem is expressed as

Minimize: \( \phi(\rho) = \frac{1}{2} \mathbf{v}^T \mathbf{Kv} \).

Subject to: \( \sum_{e=1}^{N_e} \rho_e \mathbf{v}_e \leq \mathbf{v}^* \),

\( \mathbf{Kv} = \mathbf{f} \),

(16a)

(16b)

(16c)

(16d)

(16e)

Note that in particular Eq.(5c) and Eq.(16c) are similar in form. Here \( \mathbf{v} \) is the fluid velocity vector.

The objective function sensitivity can be derived as

\[
\frac{\partial \phi}{\partial \rho_e} = \frac{(\alpha - \bar{\alpha}) q(1+q)}{2\alpha(\rho + q)^3} \mathbf{v}_e^T \mathbf{k}_{v,v} \mathbf{v}.
\]

(17)
In Eq. (15), the continuation approach is employed with $q$ starting at 0.001 and stepping up to 0.01 and 0.1. Lower and upper limit of $\alpha$ are chosen based on the assumption illustrated in Figure 5. That is with $\mu = 1$, $\alpha = 10^3 h^{-1}$ and $\alpha = 10^4 h^{-1}$ with $h \ll 1$ and $\overline{h}/h \leq 100$. Their corresponding values are pointed out in the example problems whenever needed.

We start with the design of a diffuser. The design domain and boundary conditions of a diffuser problem are illustrated in Figure 6(left). A parabolic velocity profile with unit velocity at the midstream is prescribed at the inlet while straight horizontal flow at the outlet i.e. velocity vertical component is zero. The full domain is discretized with $80 \times 80$ P1-nonconforming elements, $\kappa = 1 \times 10^{5}$, $\alpha = 40$ and $\overline{\alpha} = 40 \times 10^{4}$. The volume is constrained below 50% of the design domain. Figure 6(right) shows the optimized diffuser with flow streamlines that is almost the same as in [13].

![Figure 6: Topology optimization of diffuser: design domain and boundary condition (left), optimization history and optimized topology with flow streamlines (right)](image)

4. Conclusions
We have introduced P1-nonconforming finite element that is shown effective and useful for solving topology optimization of structures involving incompressible materials and design of channels for fluids in Stokes flow. Due to its locking-free property, problems related to incompressibility that causes poor convergence of the finite element solution are defeated even with the use of single field formulation, i.e., pure-displacement or velocity-only formulations. Furthermore, the linear shape function defined at the element vertices reduces the necessary degrees of freedom. Thus, the numerical cost for optimization can be reduced compared to the mixed methods or by using the high-order nonconforming elements particularly in large-sized problems such as 3D problems.

References

