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Uniqueness in linear and nonlinear topology optimization and approximate solutions

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Abstract

All reported results in literature indicate that even simple topology optimization problems, such as the compliance problem, have many local optimum points. Besides numerical instabilities associated to the mesh size in the finite element model, different algorithms and initial designs lead to different topologies. This phenomenon is referred to as non-uniqueness. This paper studies the conditions under which linear and nonlinear compliance problems have a unique solution, providing practical insights on the conditions that lead to non-uniqueness. To this end, three different optimization algorithms are used: OC-SIMP, SQP, and a new control-based optimization (CBO) method. The latter one has been developed to solve for nonlinear compliance problems, but it shows its best performance when solving for problems involving uniform distribution, e.g., uniform strain energy density distribution (USEDD). The solution obtained with the USEDD criterion are approximate solutions to the ones with minimum compliance; however, they can be easily obtained with the CBO method without sensitivity analysis. Furthermore, the results of this investigation show the approximate solutions closely match the exact solutions of the compliance problem.

Keywords: Control-based optimization (CBO); Sequential quadratic programming (SQP); Optimality criteria; Solid isotropic material with penalization (SIMP); Uniform strain energy density distribution (USEDD); Hybrid cellular automata (HCA); Nonlinear finite element analysis

1 Introduction

1.1 Problem formulation

A topology optimization problem can be defined as a binary programming problem in which the objective is to find the distribution of material in a prescribed area or volume referred to as the *design domain*. A classical formulation, referred to as the *binary compliance problem*, is to find the "black and white" layout (i.e., solids and voids) that minimizes mass and minimizes the work done by external forces or compliance. This problem can be formulated in two ways. One way is to define a single scalar function described as the weighted sum of the two objectives: mass and compliance. The other way is to designate one of the objectives as the primary objective function and constrain the value of the other function; for example, minimize compliance subject to a mass constraint. In either case, the nature of the problem is the same.

The binary compliance problem is known to be ill-posed (Kohn & Strang, 1986). One alternative to make the compliance problem well-posed is to control the perimeter of the structure (Haber et al., 1996; Jog, 2002). Another alternative is to relax the binary condition and include intermediate material densities in the problem formulation. This method is referred to as *homogenization method* for topology optimization (Bendsøe, 1995). The main drawback of this approach is that the optimal microstructure, which is required in the derivation of the relaxed problem, is not always known. This can be alleviated by restricting the method to a subclass of microstructures, possibly suboptimal but fully explicit. This approach, referred to as *partial relaxation* (Bendsøe & Kikuchi, 1988; Allaire & Kohn, 1993). Another problem with the homogenization methods is the manufacturability of the optimized structure ("gray" areas). However, this problem can be mitigated with *penalization* strategies. One approach is to impose *a priori* restrictions on the microstructure that implicitly lead to black-and-white designs (Bendsøe, 1995).

Even though penalization methods have shown to be effective in avoiding or mitigating intermediate densities, they revert the problem back to the original ill-possedness with respect to the mesh refinement.

An alternative that avoids the application of homogenization theory is to relax the binary problem using a continuous density value with no microstructure. The mechanical properties of the material are determined using a power-law interpolation function between void and solid (Bendsøe, 1989; Mlejnek, 1992). This power law *implicitly* penalizes intermediate density values driving the structure towards a black-and-white configuration. This approach is usually referred to as the *solid isotropic material with penalization* (SIMP) method. The SIMP method does not solve the problem's ill-possedness, but it is simpler than other penalization methods.

This work makes use of a power-law interpolation function applicable to nonlinear finite element analysis. Other approaches have considered bilinear elastoplastic materials and applied interpolation schemes for the elastic modulus, yield stress, and hardening modulus (Yoon & Kim, 2007; Patel et al., 2009). In this work, a simple interpolation function is applied to the vector of internal forces, which makes this approach more general applicable to linear and nonlinear models.

1.2 Solution methods

The *relaxed compliance problem* can be solved by a greater variety of optimization methods. Further more, when the sensitivities can be expressed in a closed form, virtually any gradient-based optimization method can be effectively used. General optimization approaches such as sequential linear programming (SLP) and sequential quadratic programming (SQP) have been extensively used. Some of the most cited approaches in include general structural optimization algorithms such as the method of moving asymptotes (MMA) and the convex linearization method (CONLIN). Besides these general approaches, there are several specialized algorithms tailored to solve particular topology optimization problems. One example is the OC-SIMP, which combines the optimality criteria (OC) of the relaxed compliance problem with the SIMP model. This approach is presented as a 99-line Matlab code (Sigmund, 2001). Due to its public availability, it is commonly used as a benchmark for other algorithms.

There is a family of heuristic methods in which the iterative scheme for topology optimization can be defined in terms of local interactions between neighboring elements. These methods are referred to as *cellular automaton* (CA) methods for topology optimization (Missoum et al., 2005; Tovar et al., 2006, 2007). The hybrid cellular automaton (HCA) method presented by Tovar et al. (2006) and Tovar et al. (2007) combines traditional finite element analysis with a local element iteration on a multivariate control system scheme. The HCA method was originally developed to solve the relaxed compliance problem. To this end, two formulations were implemented. The first makes use of a heuristic approach in which element strain energy is uniformly distributed along the design domain (Tovar et al., 2006). The final structures are free of numerical instabilities and convergence is achieved after a few function evaluations. The second formulation, referred to as HCA-SIMP, employs Karush-Kuhn-Tucker conditions with the SIMP model (Tovar et al., 2007). The HCA method has been implemented by Livermore Software Technology Corporation as LS-OPT/Topology for LS-DYNA users.

1.3 About of this work

This work studies the multiple solutions of the relaxed compliance problem. For a given length-scale, which is dictated by the mesh size of the numerical model, the relaxed compliance problem is closed and continuous, so a solution is guaranteed to exist. However, even when all conditions in the problem statement remain constant, it is known that small variations in one of the multiple parameters involved in the solution affect the final topology (Kutyłowski, 2002). The results of this work demonstrate the problem's convexity when a linear interpolation function is utilized. In the same way, this study suggests that retaining the benefits of an implicit penalization scheme jeopardizes the possibility of locating a global optimum design. Good optimization practices, such as the *continuation* method with filtering techniques, lead to a mesh-independent solution not guaranteed to be global. Furthermore, the results of such approaches depend on the optimization algorithm. The contributions of this work include the following:

- This work revises the control approach presented by Tovar et al. (2006) and restates it as a more general structural optimization algorithm for linear and nonlinear problems. This more general approach is referred to as the *control-based optimization* (CBO) method. The results of the CBO method are verified with the SQP method implemented in Matlab's optimization toolbox, and the OC-SIMP method.

- This work derives the sensitivity coefficients for the nonlinear relaxed compliance problem using a simple interpolation function. This interpolation function operates as a power law on the internal force vector in the finite element analysis. The linear version of this penalization approach is equivalent to the SIMP method.
- The results of this study show the multiple solutions for the compliance problems through test problems. This work gives insights about methodologies that lead to a mesh-independent, unique solution.

This paper is organized into three parts. The first one presents the sensitivity analysis of the compliance problem (liner and nonlinear formulations) and the derivation of the CBO method. The second part concerns the uniqueness of the compliance problem. The third one shows the application of the CBO method to an approximate formulation of the compliance problem.

2 Topology optimization problem and KKT conditions

2.1 Problem statement

In the context of the density approach, let us define the design variables x as a set n of normalized parameters $x_i \in (0, 1]$ that vary from a lower limit x_i^L and 1. The multi-objective problem of minimizing compliance C and minimizing mass M can be expressed as

$$\min_{\boldsymbol{x}^L \leq \boldsymbol{x} \leq \mathbf{1}} \left[C(\boldsymbol{x}), M(\boldsymbol{x})\right], \tag{1}$$

where $x \in \mathbb{R}^n$. Since C and M represent two conflicting functions, let us state this problem in two different forms: constrained-objective formulation and weighted-sum formulation. The constrained-objective formulation can be written as

$$\begin{array}{ll} \min\limits_{\boldsymbol{x}^L \leq \boldsymbol{x} \leq \mathbf{1}} & C(\boldsymbol{x})/C_0 \\ \text{subject to} & M(\boldsymbol{x})/M_0 - M_f = 0, \end{array} \tag{2}$$

where M_f represents the constraint limit on the mass fraction and $0 \leq M_f \leq 1$. Here again, a particular value of M_f generates a Pareto point of (1). On the other hand, the weighted-sum formulation can be expressed as

$$\min_{\boldsymbol{x}^L \leq \boldsymbol{x} \leq \mathbf{1}} \omega \frac{C(\boldsymbol{x})}{C_0} + (1 - \omega) \frac{M(\boldsymbol{x})}{M_0}, \tag{3}$$

where ω is a weighting parameter and $0 \leq \omega \leq 1$. The normalization values C_0 and M_0 are the compliance and mass of the solid structure. For each value of ω the result is a Pareto optimal point of (1).

Before presenting the optimality conditions associated to the above optimization problems, let use derive the sensitivities of the mass and the compliance with respect to the design variables for the linear and nonlinear cases.

2.2 Sensitivity analysis

The mass M of the structure is the sum of all elemental masses m_i. If the mass of a solid element is defined as m_{i0}, the $m_i = x_i m_{i0}$, where x_i is the element relative density. The mass of the structure can be defined as $M(x) = \sum_{i=1}^n x_i m_{i0}$. Therefore, the sensitivity of M with respect to x_i can be expressed as

$$\frac{\partial M}{\partial x_i} = m_{i0}, \quad \text{for } i = 1, \ldots, n. \tag{4}$$

The compliance C of the structure is the scalar quantity defined by $C(\boldsymbol{x}) = \boldsymbol{F}_{\text{ext}}^{\text{T}} \boldsymbol{U}(\boldsymbol{x})$, where $\boldsymbol{F}_{\text{ext}}$ is the applied force vector and U is the nodal displacement vector. The sensitivity of C with respect to x_i can be determined using the adjoint method and expressed as

$$\frac{\partial C}{\partial x_i} = -\boldsymbol{F}_{\text{ext}}^{\text{T}} K_T^{-1} \frac{\partial \boldsymbol{F}_{\text{int}}}{\partial x_i}. \tag{5}$$

Using the assembly operator $\mathbb{A}$, one can use the power law parameterization presented by (Khandelwal & Tovar, 2010), which can be expressed as

$$\boldsymbol{F}_{\text{int}} = \mathop{\mathbb{A}}_{i=1}^{n} \boldsymbol{f}_i^{\text{int}} = \mathop{\mathbb{A}}_{i=1}^{n} x_i^p \boldsymbol{f}_{i0}^{\text{int}},$$

where $\boldsymbol{f}_i^{\text{int}}$ is the element internal force vector and $\boldsymbol{f}_{i0}^{\text{int}}$ is the element internal force vector of a solid element. Therefore, (5) can be written as

$$\frac{\partial C}{\partial x_i} = -p\boldsymbol{F}_{\text{ext}}^{\text{T}} K_T^{-1} \left[\mathop{\mathbb{A}}_{j=1}^{n} \delta_{ij} x_j^{p-1} \boldsymbol{f}_{j0}^{\text{int}} \right], \tag{6}$$

where δ_{ij} is Kronecker's delta.

Remark 1: For a linear elastic material behavior, $K_T = K$, and $\boldsymbol{F}_{\text{ext}} = \boldsymbol{F}_{\text{int}} = K\boldsymbol{U}$, where K is the global stiffness (symmetric) matrix. Consequently, $\boldsymbol{F}_{\text{ext}}^{\text{T}} K_T^{-1} = \boldsymbol{U}^{\text{T}}$. Furthermore, $\boldsymbol{f}_i^{\text{int}} = k_i \boldsymbol{u}_i$, where k_i is the element stiffness matrix and $\boldsymbol{u}_i$ is the element nodal displacement vector. Using the power law model $\boldsymbol{f}_i^{\text{int}} = x_i^p \boldsymbol{f}_{i0}^{\text{int}} = x_i^p k_{i0} \boldsymbol{u}_i$, where k_{i0} is the stiffness matrix of a solid element, yields

$$\frac{\partial \boldsymbol{F}_{\text{int}}}{\partial x_i} = p \left[\mathop{\mathbb{A}}_{j=1}^{n} \delta_{ij} x_j^{p-1} k_{i0} \boldsymbol{u}_j \right].$$

Finally, for an linear elastic material, the sensitivity of C with respect to x_i can be expressed as

$$\frac{\partial C}{\partial x_i} = -p \frac{c_i}{x_i}, \tag{7}$$

where c_i is the element compliance given by $c_i = \boldsymbol{u}_i{}^{\text{T}} x^p k_{i0} \boldsymbol{u}_i$.

2.3 Optimality conditions

2.3.1 Case (a)

For $x_i^L < x_i < 1$, the complementary conditions are satisfied only if $\lambda_i^L = \lambda_i^U = 0$. Using the sensitivities (4) and (5), the optimality condition for the constrained-objective formulation (2) can be expressed as

$$-\frac{p}{C_0} \boldsymbol{F}_{\text{ext}}^{\text{T}} K_T^{-1} \left[\mathop{\mathbb{A}}_{j=1}^{n} \delta_{ij} x_j^{p-1} \boldsymbol{f}_{j0}^{\text{int}} \right] + \frac{\mu^*}{M_0} m_{i0} = 0. \tag{8}$$

This condition can be also expressed as $y_i = y_i^*$, where the *field variable* y_i is

$$y_i = \boldsymbol{F}_{\text{ext}}^{\text{T}} K_T^{-1} \left[\mathop{\mathbb{A}}_{j=1}^{n} \delta_{ij} x_j^{p-1} \boldsymbol{f}_{j0}^{\text{int}} \right] \tag{9}$$

and the *set point* s_i is

$$s_i = \frac{m_{i0}}{p} \frac{C_0}{M_0} \mu^*. \tag{10}$$

2.3.2 Case (b)

For $x_i = x_i^L$ (weak or void element), one obtains that $\lambda_i^U = 0$. Then the KKT conditions are satisfied when

$$-\frac{p}{C_0} \boldsymbol{F}_{\text{ext}}^{\text{T}} K_T^{-1} \left[\mathop{\mathbb{A}}_{j=1}^{n} \delta_{ij} x_j^{p-1} \boldsymbol{f}_{j0}^{\text{int}} \right] + \frac{\mu^*}{M_0} m_{i0} = \lambda_i^{*L} \geq 0,$$

or simply $y_i \leq s_i$. This conditions applies for both constrained and weighted-sum formulation and also for the linear case.

2.3.3 Case (c)

For $x_i = 1$ (solid element), $\lambda_i^0 = 0$ and the KKT conditions are satisfied when

$$-\frac{p}{C_0} \boldsymbol{F}_{\text{ext}}^{\text{T}} K_T^{-1} \left[\mathop{\mathbb{A}}_{j=1}^{n} \delta_{ij} x_j^{p-1} \boldsymbol{f}_{j0}^{\text{int}} \right] + \frac{\mu^*}{M_0} m_{i0} = -\lambda_i^{*0} \leq 0,$$

or simply $y_i \geq s_i$. As in the previous case, this conditions applies for constrained and weighted-sum formulation and for the linear case.

3 Control-based topology optimization method

3.1 Design variable update

Let us define the error signal e_i as the difference between the field variable y_i and the set point s_i. At the k-th iteration, the error signal is

$$e_i^k = y_i^k - s_i^k. \tag{11}$$

For an interior point the KKT conditions are satisfied when $e_i^* = 0$. Let us consider the following iterative scheme:

$$x_i^{k+1} = \min\left\{\max\left\{x_i^L, x_i^k + \Delta x_i^k\right\}, 1\right\}, \tag{12}$$

where the local change in the design variable Δx_i^k is a function local error signal e_i^k expressed by (11). Let the lower limit for the design variable $x_i^L = 1 \times 10^{-4}$.

3.2 Multivariable feedback control

The objective of the controller is to vanish the error $\boldsymbol{e}^k$ between the field variable $\boldsymbol{y}^k$ and the set point $\boldsymbol{s}^k$. To this end, the controller determines the change in design variable $\boldsymbol{x}^k$. The finite element analysis operates over the whole design domain and determines the field variable distribution $\boldsymbol{y}^k$ (Fig. 1).

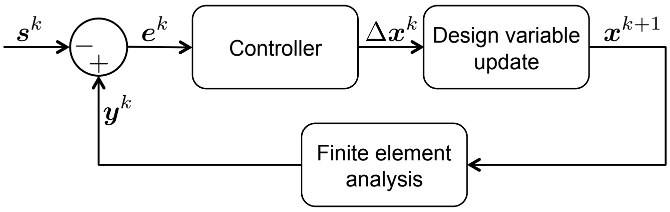

Figure 1: Multivariable feedback control of the discrete system operating simultaneously in the whole design domain.

This paper considers the case of a proportional controller and introduces a more general theory. To this end, let us consider the control-based optimization algorithm acting simultaneously in all the elements of the discretized design domain. Using a proportional controller, the change in the vector of design variables can be expressed as

$$\Delta\boldsymbol{x}^k = K_p\boldsymbol{e}^k + K_i\left(\boldsymbol{e}^0 + \boldsymbol{e}^1 + \cdots + \boldsymbol{e}^k\right) + K_d\left(\boldsymbol{e}^k - \boldsymbol{e}^{k-1}\right), \tag{13}$$

where $\boldsymbol{e}^k = \boldsymbol{y}^k - \boldsymbol{s}^k$ and K_p, K_i, and K_d are referred to as the proportional, integral, and derivative gain matrices, respectively. The controller tuning involves the selection of the optimum values of K_{pi} in order to achieve a the maximum rate of convergence, which is an active research field (Penninger et al., 2010). In order to improve the order of convergence, this paper makes use of a heuristic tuning technique in which a parameter γ_i is set for every element as a fraction of the design variable x_i. For several problems, the relation $\gamma_i^k = \frac{1}{2p}\left(x_i^k\right)^{1/p}$ resulted in a fast convergent solution.

3.3 Feasibility update

This control strategy shown in Fig. 1 is suitable when the set point remains constant throughout the iterative process, which is the case of the weighted-sum formulation (3). However, for the constrained-objective formulation (2) the set point is a function of the Lagrange multiplier μ associated to the mass constraint. In the iteration process, the constraint is satisfied when $M(\boldsymbol{x}^{k+1}) - M_f M_0 = 0$. From (12) and (13) one can show that $\boldsymbol{x}^{k+1}$ is a function of $\boldsymbol{s}^k$, then the mass M is an implicit function of the set point vector $\boldsymbol{s}^k$.

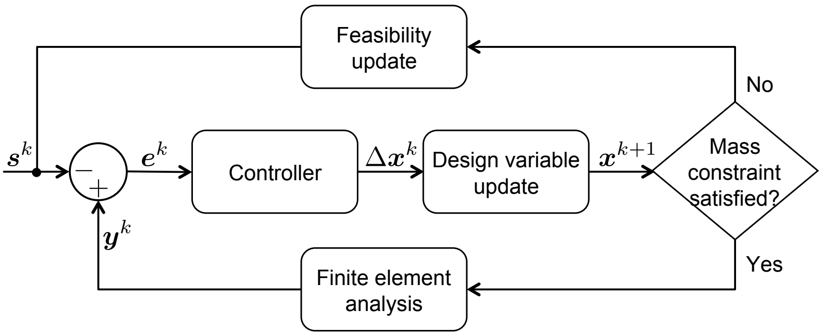

Figure 2: Multivariable control system with a feasibility update sub-loop for mass constraint.

3.4 Termination criteria

Assuming that the algorithm (12) is a contraction, a suitable criterion, based on the change in the state of the cells, can be expressed as

$$\max_{i\in\{1,\ldots,n\}} |x_i^k - x_i^{k+1}| \le \epsilon_x, \tag{14}$$

where ϵ_x is a small positive number. Once the termination criterion (14) is satisfied, the system is considered to be in steady state and no further changes in the design variables are required. In order to accurately compare the results of the CBO algorithm with other optimization techniques, let $\epsilon_x = 1\times10^{-6}$.

The algorithm checks for successive reductions in the objective function in order for stopping. Therefore, it includes a termination criterion for the normalized compliance described as

$$|C(\boldsymbol{x}^{k+1}) - C(\boldsymbol{x}^k)| \le \epsilon_c C_0, \tag{15}$$

where ϵ_c is a small positive number. In the same way, the termination criterion based on the successive change of mass can be expressed as

$$|M(\boldsymbol{x}^{k+1}) - M(\boldsymbol{x}^k)| \le \epsilon_m M_0, \tag{16}$$

where ϵ_m is a small positive number. Let $\epsilon_c = \epsilon_m = 1 \times 10^{-6}$. The three criteria are required to be satisfied for the CBO to terminate.

4 Unique and non-unique solutions

4.1 Test problem

Let us consider a continuum structure in cantilever of dimensions $2L\times L$ and thickness h with a downward constant force F applied in the lower free node as depicted in Fig. 3.

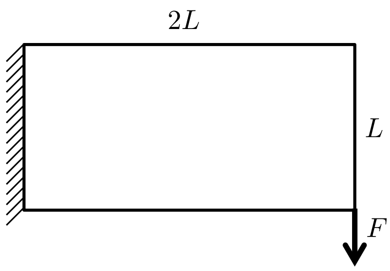

Figure 3: Plate in cantilever of dimensions $2L \times L \times h$ and a constant force F.

The material of the structure is steel with a Young's modulus $E = 200$ GPa and Poisson's ratio $\nu = 0.3$. Let $F = 10$ kN and the dimensions of the plate 200 mm × 100 mm × 10 mm. Using the constrained-objective formulation (2) with $M_f = 0.5$ the optimization problem can be stated as

$$\begin{aligned} \min_{\boldsymbol{x}^L \le \boldsymbol{x} \le \boldsymbol{1}} \quad & C(\boldsymbol{x})/C_0 \\ \text{subject to} \quad & M(\boldsymbol{x})/M_0 - 0.5 = 0, \end{aligned} \tag{17}$$

where $x_i^L = 1 \times 10^{-4}$ for $i = 1, \ldots, n$. The volume of the design domain is $V_0 = 200$ cm^3. For a density of $\rho_0 = 8$ g/cm^3, the mass of the solid design domain is $M_0 = 1.6$ kg. Therefore, the target final mass is 800 g. The compliance of the solid structure C_0 will be determined by FEA. Quadrilateral bilinear finite elements are used under plane stress condition.

4.2 Numerical methods

This compliance problem will be solved using the CBO algorithm as described in Section **??**. The results will be compared to the ones obtained by the well-established SQP algorithm. The same termination criteria are used in these two numerical approaches. The SQP method makes use of the closed form expression for the sensitivities of the compliance with respect to the design variables. The mass is imposed as a linear constraint. A positive definite quasi-Newton approximation of the Hessian of the Lagrangian function is calculated using the BFGS method. The QP sub-problem is solved using an active set strategy (or projection method). The solution to the QP sub-problem produces a search direction to perform a line search and updated the design. This algorithm is incorporated in Matlab's optimization toolbox by the function `fmincon()` with the option `'sqp'`. The OC-SIMP method implemented in Matlab by Sigmund (2001) follows the heuristic updating scheme proposed by Bendsøe (1995). This method involves a move limit and a damping coefficient with default values 0.2 and 0.5, respectively. It also makes use of a sensitivity filtering function, which is not used in this study. The termination criteria for SQP and OC-SIMP are set as the ones used by HCA in Section 3.4.

4.3 Two-element problem

The purpose of this study is to show the application of the CBO and compare the result to the problem's analytic solution and the one using SQP. To this end, let us discretized the design domain in two identical elements elements of dimensions 100 mm $\times$ 100 mm $\times$ 10 mm. For linear elastic analysis, the compliance for $x_1 = 1$ and $x_2 = 1$ is $C_0 = 1306.6$ Nmm for any value of the power p.

For $p = 1$ (thickness optimization), the analytical solution is $x_1^* = 0.664752$ and $x_2^* = 0.335248$ with a relative compliance $C/C_0 = 1.80440$. Starting from the initial design $x_1^0 = 0.5$ and $x_2^0 = 0.5$, the CBO algorithm converges to the exact solution after 5 function evaluations. The SQP algorithm converges in 6 iterations using 15 function. The OC-SIMP method requires only 3 function evaluations.

For $p = 3$ (density optimization), the analytical solution is $x_1^* = 0.584801$ and $x_2^* = 0.415199$ with a relative compliance $C/C_0 = 6.81990$. The CBO converges after 6 FE analyses; here again, one might observe convergence under 4 iterations with the proper adjustment of the proportional gain. The SQP algorithm converges in 5 iterations using 15 FE analyses. The OC-SIMP converges after 2250 iterations.

4.4 Eight-element problem

Let us discretize the design domain into eight identical squared elements of dimensions 50 mm $\times$ 50 mm $\times$ 10 mm. For $\boldsymbol{x} = \mathbf{1}$, the compliance is $C_0 = 1728.63$ Nmm for any value of p. For $p = 1$, the solution can be expressed as

$$\boldsymbol{x}^* = \begin{pmatrix} 0.75226 & 0.57188 & 0.38537 & 0.17127 \\ 0.74933 & 0.56212 & 0.37748 & 0.43031 \end{pmatrix},$$

with a relative compliance of $C/C_0 = 1.76500$. From the initial design $x_i^0 = 0.5$ for $i = 1, \ldots, 8$, the CBO converges to this solution after 11 function evaluations. The SQP algorithm converges in 19 iterations using 85 function evaluations. The OC-SIMP algorithm convergence to the solution after 12 function evaluations. The same solution is found from any starting point.

For $p = 3$, let us use the solution for $p = 1$ as the initial design. This approach is somewhat similar to one used by the continuation method. In this case, all algorithms converged to the same stationary point

$$\boldsymbol{x}^* = \begin{pmatrix} 0.64716 & 0.56971 & 0.50979 & 0.00010 \\ 0.63729 & 0.55141 & 0.51205 & 0.57249 \end{pmatrix},$$

with relative compliance of $C/C_0 = 5.86492$. The CBO algorithm used 25 function evaluations, SQP used 18 iterations and 82 function evaluations, and OC-SIMP required 689 function evaluations.

The same solution is obtained using from the initial point $x_i = 0.5$ for $i = 1, \ldots, 8$ using HCA and OC-SIMP, but SQP converges to a different stationary point. More generally, different initial designs lead to different solutions.

4.5 Two-hundred element problem

Let us discretize the design domain into 200 identical elements of size 10 mm $\times$ 10 mm $\times$ 10 mm. For $\boldsymbol{x} = \mathbf{1}$ and linear analysis, the compliance is $C_0 = 2124.91$ Nmm for any value of p. For $p = 1$ there is a unique solution and for $p = 3$ there are several stationary points. From the initial design $x_i^0 = 0.5$ for $i = 1, \ldots, 200$ and $p = 1$, the CBO converges after 344 function evaluations (Fig. 4). The SQP algorithm converges to the same solution after 258 iterations using 518 FE analyses. The OC-SIMP uses 346 iterations. It is worth noticing that the termination criteria used in this application and, therefore, the resulting number of function evaluations, are academically useful. However, for a practical application the tolerances must be significantly reduced.

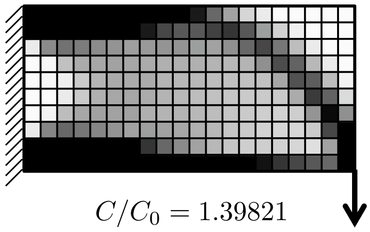

Figure 4: Optimum design for $p = 1$.

When using a power $p = 3$ there are many fixed points. These points can be found by slightly changing the initial designs. As an illustration, let us use the optimum design for $p = 1$ as the initial design. The results for each one of the three algorithms are shown in Fig. 5. The CBO algorithm converges to a solution of relative compliance $C/C_0 = 1.63131$ after 99 function evaluations. The SQP finds a design of $C/C_0 = 1.66845$ using 170 iterations and 341 function evaluations. The OC-SIMP algorithm converges to a design of $C/C_0 = 1.66044$ after 3264 function evaluations. These solutions are only a small sample of hundreds of stationary points for these algorithms. In particular, these three designs depict checkerboard patterns and a few elements with intermediate densities.

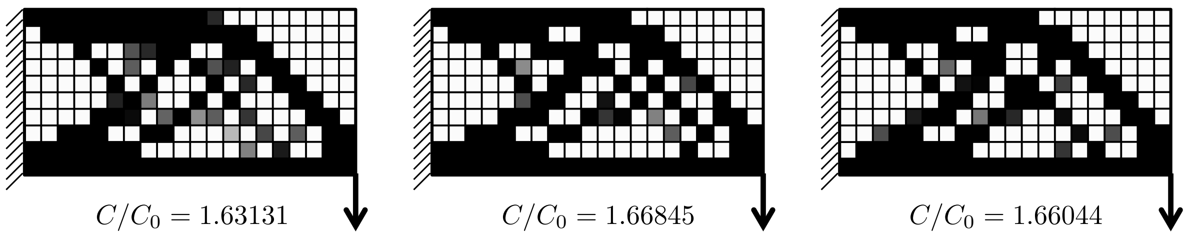

Figure 5: Local optimum points for $p = 3$ found with CBO (left), SQP (center), and OC-SIMP (right). The result for $p = 1$ was use was used as initial design in every case.

5 Uniform strain energy distribution problem

An approximation to the compliance is the uniform strain energy density distribution (USEDD) problem. This problem can be expressed as finding the material distribution that minimizes the norm of element compliances. In other words, the variation of the element compliance with respect to a zero value. This can be written as

$$\begin{aligned} \min_{\boldsymbol{x}^L \leq \boldsymbol{x} \leq \mathbf{1}} \quad & \|\boldsymbol{c}(\boldsymbol{x})\| \\ \text{subject to} \quad & M(\boldsymbol{x})/M_0 - M_f = 0, \end{aligned} \tag{18}$$

where $\boldsymbol{c}$ is the vector of element compliances, which is a non-negative quantity. In the the linear case, $c_i = \boldsymbol{u}_i{}^{\mathrm{T}} x^p \boldsymbol{k}_{i0} \boldsymbol{u}_i$. Using norm infinity, (18) can be expressed as

$$\begin{aligned} \min_{\boldsymbol{x}^L \leq \boldsymbol{x} \leq \mathbf{1}} \max_{i \in \{1, \ldots, n\}} \quad & c_i(\boldsymbol{x}) \\ \text{subject to} \quad & M(\boldsymbol{x})/M_0 - M_f = 0. \end{aligned} \tag{19}$$

As observed by the reader, the sensitivity analysis of (19) is rather simple. Using $p = 1$ and the initial design $x_i = 0.5$ for $i = 1, \ldots, 8$ (eight-element problem), SQP finds the solution using 7 iterations and 87 function evaluations. This solution can be expressed as

$$\boldsymbol{x}^* = \begin{pmatrix} 1.00000 & 0.57987 & 0.26068 & 0.09744 \\ 0.95550 & 0.53323 & 0.24476 & 0.32846 \end{pmatrix},$$

with a relative compliance of $C/C_0 = 1.905859$, which is larger than the one obtained for the compliance problem. The vector of element compliance for the optimum design is

$$\boldsymbol{c}(\boldsymbol{x}^*) = \begin{pmatrix} 4.42488 & 4.42488 & 4.42488 & 2.01026 \\ 4.42488 & 4.38570 & 4.42488 & 4.42489 \end{pmatrix} \times 10^2 \text{ Nmm}.$$

The same solution is obtained from any (random) initial design. Even though, there is no improvement in the compliance nor the number of function evaluations for SQP, the advantage of this formulation is the simple sensitivity analysis.

6 Approximate USEDD problem

The CBO method presented in Section 3 requires the definition of a field variable y_i^k and a set point s_i^k. The field variable and set points have been defined from KKT conditions. However, let us use a heuristic approach in which the strain energy density (or element compliance) is used as the field variable. This formulation shares similarities to the USEDD method as the deviation between the SED and a constant set point is locally minimized. The optimization problem that is locally solved can be expressed as

$$\min_{x_i^L \le x_i \le 1} \quad |y_i^k - s_i^k|, \tag{20}$$

where $y_i^k = c_i^k$ (element compliance) and the set point s_i^k is the same for all $i = 1, \ldots, n$. If the mass of the structure is constrained, then the set point is determined via the feasibility loop and it varies at every iteration k; otherwise, it remains constant for all k. No sensitivity analysis is performed. Using $p = 1$ and the initial design $x_i = 0.5$ for $i = 1, \ldots, 8$ (eight-element problem), CBO find the solution using 57 function evaluations. The stationary point found can be expressed as

$$\boldsymbol{x}^* = \begin{pmatrix} 0.93546 & 0.55924 & 0.34725 & 0.00010 \\ 0.81252 & 0.45168 & 0.34428 & 0.54947 \end{pmatrix},$$

with a relative compliance of $C/C_0 = 2.01129$, which is larger than the one from the compliance and the USEDD problems. The vector of element compliance of the optimum design has a rather uniform distribution, which is

$$\boldsymbol{c}(\boldsymbol{x}^*) = \begin{pmatrix} 4.96524 & 4.96524 & 4.96524 & 0.01125 \\ 4.96522 & 4.96521 & 4.96521 & 4.96523 \end{pmatrix} \times 10^2 \text{ Nmm}.$$

In contrast to the USEDD, different stationary points are obtained from different initial designs. In a sense, for $p = 1$ the results are similar to the ones obtained using a penalized ($p > 2$) USEDD formulation. The real advantage of this method is that no sensitivity analysis is required. This method can be easily incorporated to any commercial nonlinear finite element program.

7 Conclusions

The problem concerning this paper is the existence of multiple solutions for a fixed mesh size with the same penalization method. To this end, a simple and computationally efficient control-based optimization(CBO) method is proposed. This method is founded on the basis of the uniform distribution of a field variable over the structure's design domain (Tovar et al., 2006). In the CBO method presented in this work, one controller is assigned to each finite element. Each controller minimizes the error between a set point an the field variable. A local actuator modifies the density of the finite element according to the error signal. The expression for the the set point and the field variable can be obtained from the Karush-Kuhn-Tucker conditions of the optimization problem (Tovar et al., 2007). In this work, these expressions are derived from a fully nonlinear topology optimization problem. The controllers of the

CBO method are adaptive. This method has been used to solve the linear compliance problem and the results were verified with the SQP method and the OC-SIMP method. The CBO reaches convergence in considerably fewer iterations than the other methods. This paper presents the results when uniform strain energy density distribution (USEDD) criterion is applied. This problem is rather simple to solve and leads to similar results than the ones from the compliance problem. Finally, an approximate USEDD problem is formulated. This problem does not require sensitivity analysis and leads to similar results than the other two methods. The advantage of the approximate USEDD formulation is that it can be readily used with any nonlinear finite element software.

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