Optimization of $k$-$\varepsilon$ turbulence models for incompressible flow around airfoils

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Abstract
In this paper a simple optimization is carried out for a widely known $k$-$\varepsilon$ turbulence model by using a random search to find improved sets of closure coefficients for this specific type of flow. This preliminary work tries to show whether the optimization of these parameters also implies an improvement for different airfoils. The results show that this hypothesis is valid.

Keywords: Optimization, turbulence model, closure coefficients, finite element analysis.

1. Introduction
For turbulent flows with very high Reynolds number the ratio between the lengths of the largest and smallest eddies is often around a million, and because of that, the Navier-Stokes Direct Numerical Simulation (DNS) method will probably remain unaffordable for decades [1] so turbulence models will remain essential. Since the 1970s the most commonly used turbulence models for high Reynolds number flows are the two equation $k$-$\varepsilon$ models [2,3,4,5], or combined models [6] that use them in high Reynolds number flow regions. These models close the RANS (Reynolds Averaged Navier-Stokes) system of equations that governs the mean flow by approximating the effect of turbulence on it by using a turbulent viscosity that varies with space and time as a function of the kinetic energy of the turbulence $k$ and its dissipation rate $\varepsilon$. The transport of these two variables is approximated by two semi-empirical model equations whose coefficients are tuned by minimizing errors over a wide variety of flows [2,7]. This fact provides generality to these kinds of models, but on the other hand produce large errors for particular flows.

In the case of incompressible flows around airfoils at high Reynolds numbers, large errors in the drag force are produced. This is seen, for instance, by comparing the aerodynamic coefficients of NACA airfoils obtained with the $k$-$\varepsilon$ models on finite element meshes with wind tunnel empirical results [8]. The discussion above suggests that an improvement in the performance of the $k$-$\varepsilon$ turbulence models can be achieved for this particular type of flow by optimizing the closure coefficients. In this way, the optimization variables become the closure coefficients, and the function to minimize contains the errors of flows around a set of airfoils at several angles of attack. Afterwards, the results obtained are applied to an alternate set of airfoils to check if they are also improved. Both sets of airfoils include a wide variety of geometries. This method should improve the behaviour of the model for other similar problems [9,10].

2. Optimization of the closure coefficients for the standard $k$-$\varepsilon$ model
For incompressible flow, the mean flow Reynolds Averaged Navier-Stokes (RANS) system of equations when using two-equation standard $k$-$\varepsilon$ turbulence model can be written as follows,

$$ \frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = - \frac{P}{\rho} \frac{\partial}{\partial x_j} \left( \rho \delta_{ij} + 2(v + v_r)S_{ij} \right) $$

$$ \frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} = \left( \frac{v_k}{\sigma_k} + \frac{v_r}{\sigma_r} \right) \frac{\partial \varepsilon}{\partial x_j} + \Pi - \varepsilon $$

$$ \frac{\partial \varepsilon}{\partial t} + U_j \frac{\partial \varepsilon}{\partial x_j} = \left( \frac{v_k}{\sigma_k} + \frac{v_r}{\sigma_r} \right) \frac{\partial k}{\partial x_j} + c_1 \frac{\varepsilon}{k} - c_2 \frac{\varepsilon^2}{k} $$

where Eq.(1) is the mass conservation equation, Eq.(2) is the momentum conservation equation, being Eq.(3) and Eq.(4) the model equations for the transport of $k$ and $\varepsilon$ respectively. In these expressions $U_i$ and $P$ are the mean flow velocity and pressure respectively, $v_r=c_k\varepsilon^2/\varepsilon$ is the kinematic turbulent viscosity, being $S_{ij}$ the strain rate tensor of the mean flow, while $\Pi=2\nu_rS_{ij}S_{ij}$ is the production of turbulent kinetic energy. The last two terms in Eq.(4) are called the production and destruction of $\varepsilon$. The closure coefficients are

$$ (c_{1,2,3,4,5}) = (1.44, 1.92, 0.09, 1, 1.3) $$

In this paper our aim is to improve the accuracy of this turbulence model for high Reynolds number incompressible
aerodynamic flow around airfoils at angles of attack below the stall angle. The optimization variables are the closure parameters and the function to minimize contains the errors of flows around a set of airfoils at several angles of attack. Afterwards the results are applied to an alternative set of airfoils to check if they also provide improvement. Both sets of airfoils include a wide variety of geometries.

2.1. The evaluation function \( \Phi \)

Sets of closure parameters that produce results more closely related to wind tunnel experiments than the original ones will be searched by minimizing the evaluation function denoted by \( \Phi \). This function is a weighted sum of the squares of the errors obtained for the aerodynamic coefficients \( c_i \), \( c_d \) and \( c_m \) in a group of flows at Reynolds number \( Re=6\cdot10^6 \). These are the flows around airfoils naca66-206, naca0012 and naca4424 for the three respective angles of attack (-4,0,6), (0,4,8) and (-4,0,6). As can be seen in figure 1, these airfoils have very different geometries.

![Fig1: Airfoils for the evaluation function \( \Phi \).](image)

The evaluation function, in terms of the relative errors, is written as

\[
\Phi = \sum_{i=1}^{3} \sum_{j=1}^{3} \left[ W_i c_i \xi_i \right]^2 + \sum_{i=1}^{3} \sum_{j=1}^{3} \left[ W_i c_d \xi_d \right]^2 + \sum_{i=1}^{3} \sum_{j=1}^{3} \left[ W_i c_m \xi_m \right]^2
\]

(6)

where \( i \) indicates the airfoil, \( j \) the angle of attack and \( \xi_{ci} \), \( \xi_{cd} \) and \( \xi_{cm} \) the relative errors for the coefficients \( c_i \), \( c_d \) and \( c_m \), defined as follows

\[
\xi_i = \frac{c_i^{\text{model}} - c_i^{\text{exp}}}{c_i^{\text{exp}}}, \quad \xi_d = \frac{c_d^{\text{model}} - c_d^{\text{exp}}}{c_d^{\text{exp}}}, \quad \xi_m = \frac{c_m^{\text{model}} - c_m^{\text{exp}}}{c_m^{\text{exp}}}
\]

(7)

being \( c_i^{\text{model}} \) the aerodynamic coefficients produced by the turbulence model and \( c_i^{\text{exp}} \) the corresponding wind tunnel experimental results [8]. The constants \( W_i c_i \), \( W_d c_d \) and \( W_m c_m \) weigh the importance of the errors \( \xi_{ci} \), \( \xi_{cd} \) and \( \xi_{cm} \). Since, for some of these flows, the experimental values of \( c_i \), \( c_d \) or \( c_m \) are zero or very small, the value of the evaluation function will be very high and independent of the quality of the model. Thus, in order to avoid this problem, the evaluation function is defined in terms of the absolute errors, as follows:

\[
\Phi = \sum_{i=1}^{3} \sum_{j=1}^{3} \left| W_i \xi_i \right|^2 + \sum_{i=1}^{3} \sum_{j=1}^{3} \left| W_i \xi_d \right|^2 + \sum_{i=1}^{3} \sum_{j=1}^{3} \left| W_i \xi_m \right|^2
\]

(8)

where \( \xi_i \), \( \xi_d \) and \( \xi_m \) are the absolute errors for \( c_i \), \( c_d \) and \( c_m \), which are written as

\[
\xi_i = C_i^{\text{model}} - C_i^{\text{exp}}, \quad \xi_d = C_d^{\text{model}} - C_d^{\text{exp}}, \quad \xi_m = C_m^{\text{model}} - C_m^{\text{exp}}
\]

(9)

and \( W_i \), \( W_d \) and \( W_m \) are weighting constants. As the magnitudes of \( \xi_i \) are near 0.1, those of \( \xi_d \) around 0.01 and those of \( \xi_m \) in between, in order for the three terms in expression Eq.(8) to have a similar magnitude, the values 1, 10 and 5 were assigned respectively for \( W_i \), \( W_d \) and \( W_m \). This way, which considers the form of the errors of the standard \( k-\varepsilon \) model causes the biggest angles of attack to contribute the most to the value of the evaluation function.

2.2. The checking function \( \Psi \)

Good sets of closure constants scored with values of \( \Phi \) smaller than the original set Eq.(5) for the three airfoils, will afterwards be calculated for the alternative group of five airfoils of variable shape, shown in figure 2, to check if any improvement of the performance of the turbulence model is produced for them. This new group of airfoils is formed by naca63A210, naca4412, naca63,-421, naca66,-212 and naca67,1-215, and each is calculated for the respective three angles of attack (-4,0,8), (-4,0,8), (-4,0,8), (-4,0,8) and (-4,0,6). The values of the checking function denoted by \( \Psi \) are assigned to each of these sets of constants. \( \Psi \) is defined in the same form as the evaluation function \( \Phi \), as follows:

\[
\Psi = \sum_{i=1}^{3} \sum_{j=1}^{3} \left| W_i \xi_i \right|^2 + \sum_{i=1}^{3} \sum_{j=1}^{3} \left| W_i \xi_d \right|^2 + \sum_{i=1}^{3} \sum_{j=1}^{3} \left| W_i \xi_m \right|^2
\]

(10)

where \( i \) indicates the airfoil, \( j \) the angle of attack and the weighting constants \( W_i \), \( W_d \) and \( W_m \) have the same values as above.
2.3. Random search
As a preliminary exercise, an optimization of the closure coefficients of the standard k-ε model was made by random
search. The flows were calculated using Fidap finite element code on meshes of around 16,000 finite elements, similar
to that shown in figure 3. The original set of closure constants gave the value $\Phi^{\text{orig}}=0.416$ for the evaluation function,
yielding $\Psi^{\text{orig}}=0.69$ for the checking function.

The optimization variables were defined for the following intervals: $[0.77, 2.88]$ for $c_{\epsilon 1}$, $[0.96, 3.84]$ for $c_{\epsilon 2}$, $[0.02, 0.18]$ for
$c_{\mu}$, $[0.5, 2]$ for $\sigma_k$, and $[0.65, 2.6]$ for $\sigma_\epsilon$. In order to minimize the evaluation function $\Phi$, ten executions were made
over a population of 400 individuals. The best five individuals for each execution were afterwards checked with the alternate set of airfoils, assigning a value of $\Psi$ to each individual. Table 1 lists the closure constants for these 50
individuals, along with the respective values of the functions $\Phi$ and $\Psi$, and the sum of both.

Table 1: Values of the parameters and functions $\Phi$ and $\Psi$, and $\Phi+\Psi$ for the five best individuals in each random search
execution made over a population of 400 individuals.

<table>
<thead>
<tr>
<th></th>
<th>$c_{\epsilon 1}$</th>
<th>$c_{\epsilon 2}$</th>
<th>$c_{\mu}$</th>
<th>$\sigma_k$</th>
<th>$\sigma_\epsilon$</th>
<th>$\Phi$</th>
<th>$\Psi$</th>
<th>$\Phi+\Psi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>original model</td>
<td>1.44</td>
<td>1.92</td>
<td>0.09</td>
<td>1.0</td>
<td>1.3</td>
<td>0.416</td>
<td>0.690</td>
<td>1.1060</td>
</tr>
</tbody>
</table>
In order to reflect the quality of these selected models, the results of three of them denoted A, B and C, collected in five best inds.

Table 2: Three individuals whose value of the sum $\Phi + \Psi$ are respectively the first, the 25th and the 50th of the fifty runs.

<table>
<thead>
<tr>
<th>five best inds.</th>
<th>$\Phi$</th>
<th>$\Psi$</th>
<th>$\Phi + \Psi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st run</td>
<td>2.85</td>
<td>3.32</td>
<td>6.17</td>
</tr>
<tr>
<td>2nd run</td>
<td>2.37</td>
<td>3.10</td>
<td>5.47</td>
</tr>
<tr>
<td>3rd run</td>
<td>2.64</td>
<td>3.78</td>
<td>6.42</td>
</tr>
<tr>
<td>4th run</td>
<td>2.31</td>
<td>2.85</td>
<td>5.16</td>
</tr>
<tr>
<td>5th run</td>
<td>2.46</td>
<td>2.90</td>
<td>5.36</td>
</tr>
<tr>
<td>6th run</td>
<td>2.30</td>
<td>3.05</td>
<td>5.35</td>
</tr>
<tr>
<td>7th run</td>
<td>2.08</td>
<td>3.45</td>
<td>5.53</td>
</tr>
<tr>
<td>8th run</td>
<td>2.71</td>
<td>3.36</td>
<td>5.07</td>
</tr>
<tr>
<td>9th run</td>
<td>2.81</td>
<td>2.58</td>
<td>5.39</td>
</tr>
<tr>
<td>10th run</td>
<td>2.30</td>
<td>3.40</td>
<td>5.70</td>
</tr>
</tbody>
</table>

In order to reflect the quality of these selected models, the results of three of them denoted A, B and C, collected in five best inds.

Table 2: Three individuals whose value of the sum $\Phi + \Psi$ are respectively the first, the 25th and the 50th of the fifty runs.
checked individuals.

<table>
<thead>
<tr>
<th></th>
<th>$c_{l1}$</th>
<th>$c_{l2}$</th>
<th>$c_m$</th>
<th>$\sigma_{l1}$</th>
<th>$\sigma_m$</th>
<th>$\Phi$</th>
<th>$\Psi$</th>
<th>$\Phi+\Psi$</th>
<th>range $\Phi+\Psi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>model A</td>
<td>2.78</td>
<td>3.84</td>
<td>0.06</td>
<td>1.4</td>
<td>0.69</td>
<td>0.308</td>
<td>0.573</td>
<td>0.8811</td>
<td>1st</td>
</tr>
<tr>
<td>model B</td>
<td>2.56</td>
<td>3.41</td>
<td>0.05</td>
<td>1.82</td>
<td>1.65</td>
<td>0.313</td>
<td>0.587</td>
<td>0.8999</td>
<td>25th</td>
</tr>
<tr>
<td>model C</td>
<td>2.3</td>
<td>3.05</td>
<td>0.03</td>
<td>1.73</td>
<td>0.71</td>
<td>0.303</td>
<td>0.644</td>
<td>0.9465</td>
<td>50th</td>
</tr>
</tbody>
</table>

In figure 4 the values of $c_l$, $c_d$ and $c_m$ for these three models are shown for the eight airfoils used. They are compared with the results of both the original model and the wind tunnel experiments.
Fig. 4: Aerodynamic coefficients $c_l$, $c_d$ and $c_m$ for the eighth used airfoils, results for model A (---), model B (- - -), model C (.....), standard original model (____) and wind tunnel experiments (□).

It is observed that all fifty selected individuals yielded values of $c_{l1}$ and $c_{l2}$ larger than the original ones, indicating the trend of good models to increase production and destruction of $\varepsilon$ in their respective flow regions.

3. Conclusions
Well known turbulence models produce quite inaccurate results for aerodynamic loads in incompressible flows around airfoils at high Reynolds number. In order to enhance the results for this type of flow an optimization of the standard $k-\varepsilon$ turbulence model closure coefficients was performed by minimizing the errors over a group of airfoils of variable shape at various angles of attack. Ten executions of random search were done over populations of 400 individuals, and the five sets of constants obtained from each execution, which overscored the original constants, were afterwards checked over an alternative group of airfoils of very different geometry. These fifty individuals also improved the results of the original constants on the second group of airfoils, thus showing the relevance of the optimization of these parameters. This suggests that the same method can be applied to improve the analysis of other types of turbulent flow. Future work will include improvement of the optimization of different turbulence models by using more powerful methods, as genetic algorithms, hill-climbing or gradient methods.

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References