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Optimal orientation of anisotropic material with given Kelvin moduli in FMO problems for plates and shells

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Abstract
In this work we investigate the compliance minimization problem of a plate subjected to the in-plane and transverse loadings acting simultaneously. In this way we attempt to generalize the classical Free Material Optimization considerations to the coupled membrane-bending loading case. In our approach we utilize the spectral representation of a fourth-rank constitutive (stiffness) tensor in finding the optimal orientation of its second-rank eigentensors. The case under study is based on three assumptions: the plate is homogeneous with respect to its thickness, the design variables are not restricted by any isoperimetric condition and the Kelvin moduli values are kept fixed on the middle plane of a structure. Optimization task is thus reduced to the equilibrium problem of an effective hyperelastic plate with strictly convex effective nonlinear potential expressed in terms of strains. Corresponding constitutive equations are analytically and explicitly derived which allows determining all components of the optimized stiffness tensor.
Keywords: free material optimization, plates, shells

1. Introduction
Elastic properties of the three-dimensional elasticity constitutive tensor are determined by six independent moduli of stress dimension (Pa) and by fifteen nondimensional geometric independent parameters. This characterization follows from the spectral decomposition of Hooke's tensor, cf. [7], where the author proposed to attribute six elastic moduli to the name of Kelvin. The meaning of moduli and geometric parameters involved in the spectral representation is best seen, if one has an insight into the geometric properties of the RVE cells, while considering the theory of the non-homogeneous media from the mechanics of composites point of view. The theory of spectral decomposition of Hooke's tensors was also developed in e.g. [6], [8], [9].
Elastic properties of two-dimensional problems are determined by three Kelvin moduli ($\lambda_1,\ \lambda_2,\ \lambda_3$) and three geometric parameters, see [2] and [3]. In the present paper this spectral representation will be applied to describe stiffness distribution in thin elastic transversely homogeneous plate, whose model is based on the assumption of so-called generalized plane state of stress. This allows to determine both membrane and bending stiffnesses by one fourth-rank tensor $\boldsymbol{A}$ of Hooke's symmetry.
Subject of the present paper is to construct the optimal layout of geometric parameters of tensor $\boldsymbol{A}$ for the problem of simultaneous in-plane and transverse deformation of plates. The values of Kelvin moduli are kept fixed, hence the aim of the optimization is to determine the directions of corresponding eigentensors thus minimizing the compliance of the plate made from this class of materials, which means the stiffening of a plate against given loading. The problem discussed here does not necessitate imposition of any isoperimetric condition, which makes it more universal.
Optimization task considered in the sequel belongs to the free material optimization (FMO) or free material design (FMD) class of problems researched intensively since the 1990's, see e.g. [1], [4], [5]. However, the approach proposed in the present paper makes formulating the FMO problem possible using different tools, thus allowing for a deep insight into the mathematical structure of the stiffness moduli tensor. It also paves the way for the new version of FMO, with non-standard isoperimetric conditions concerning the Kelvin moduli.

2. Problem formulation
Consider a plate of continuously varying thickness $h = h(x)$, $x \in \Omega$, where Ω denotes the middle plane of a plate and assume that the material is homogeneously distributed in a direction perpendicular to Ω. Suppose that the deformation of thus defined structure is described by the theory of thin plates therefore transverse deformations are neglected while membrane and bending strain fields $\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta}(\boldsymbol{u})$,

$\varkappa_{\alpha\beta} = \varkappa_{\alpha\beta}(w)$, are defined as

$$\varepsilon_{\alpha\beta}(\boldsymbol{u}) = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}), \qquad \varkappa_{\alpha\beta}(w) = -w_{,\alpha\beta}, \tag{1}$$

where $\boldsymbol{u}(x) = (u_1(x),\ u_2(x))$ and $w(x)$ represent kinematically admissible in-plane and transverse displacement fields respectively. Moreover, $(\cdot)_{,\alpha} = \partial/\partial x_\alpha$, where the Cartesian system $(x_1,\ x_2)$ parameterize Ω. If complemented with axis x_3 orthogonal to Ω the coordinate system $(x_1,\ x_2,\ x_3)$ is assumed counterclockwise.
Let tensor $\boldsymbol{C} = (C^{\alpha\beta\lambda\mu})$ comprise all moduli of the generalized plane stress state and set

$$A^{\alpha\beta\lambda\mu} = hC^{\alpha\beta\lambda\mu}, \qquad D^{\alpha\beta\lambda\mu} = \frac{h^3}{12}C^{\alpha\beta\lambda\mu}. \tag{2}$$

Stress resultants $\boldsymbol{N} = (N^{\alpha\beta})$ and couple resultants $\boldsymbol{M} = (M^{\alpha\beta})$ are linked with strains by linear equations

$$N^{\alpha\beta} = A^{\alpha\beta\lambda\mu}\varepsilon_{\lambda\mu}, \qquad M^{\alpha\beta} = D^{\alpha\beta\lambda\mu}\varkappa_{\lambda\mu} \tag{3}$$

If the plate is subject to in-plane and transverse loadings of intensities $\boldsymbol{p}(x) = (p^1(x),\ p^2(x))$ and $q(x)$ respectively then their virtual work on the test in-plane and transverse displacements $\boldsymbol{v}(x) = (v_1(x),\ v_2(x))$ and $v(x)$ is expressed as

$$f(\boldsymbol{v},\ v) = \int_\Omega (p^\alpha v_\alpha + qv)dx. \tag{4}$$

Second-order plane symmetric tensors and fourth-order tensors of Hooke's symmetry belong to the spaces denoted by $\mathbb{E}_s^2$ and $\mathbb{E}_s^4$ respectively. Certain geometrical analogy, see [8], allows treating objects belonging to these spaces as vectors and second-order tensors in $\mathbb{R}^3$. Their representations are thus given by

$$\begin{aligned} \boldsymbol{a} &= \begin{bmatrix} a_{11} \\ a_{22} \\ \sqrt{2}a_{12} \end{bmatrix}, && \boldsymbol{a} \in \mathbb{E}_s^2, \\ \boldsymbol{A} &= \begin{bmatrix} A_{1111} & A_{1122} & \sqrt{2}A_{1112} \\ A_{1122} & A_{2222} & \sqrt{2}A_{1222} \\ \sqrt{2}A_{1112} & \sqrt{2}A_{1222} & 2\,A_{1212} \end{bmatrix}, && \boldsymbol{A} \in \mathbb{E}_s^4. \end{aligned} \tag{5}$$

Obviously, components of thus defined representations depend on the choice of basis in Ω. For brevity of further derivation define the following operations on objects from $\mathbb{E}_s^2$ and $\mathbb{E}_s^4$

$$\begin{aligned} \boldsymbol{a}\cdot\boldsymbol{b} &= \sum_{i=1}^{3} a_i\, b_i, && \boldsymbol{a} \in \mathbb{E}_s^2, \boldsymbol{b} \in \mathbb{E}_s^2, \\ \boldsymbol{A} : \boldsymbol{B} &= \sum_{i=1}^{3}\sum_{j=1}^{3} A_{ij}\, B_{ij}, && \boldsymbol{A} \in \mathbb{E}_s^4, \boldsymbol{B} \in \mathbb{E}_s^4, \\ \boldsymbol{A}\,\boldsymbol{b} &= \sum_{j=1}^{3} A_{ij}\, b_j, && \boldsymbol{A} \in \mathbb{E}_s^4, \boldsymbol{b} \in \mathbb{E}_s^2, \\ \boldsymbol{A}\,\boldsymbol{B} &= \sum_{j=1}^{3} A_{ij}\, B_{jk}, && \boldsymbol{A} \in \mathbb{E}_s^4, \boldsymbol{B} \in \mathbb{E}_s^4. \end{aligned} \tag{6}$$

The first and second equation in Eqs (6) denote scalar products in corresponding spaces. Respective norms are thus defined as $\|\boldsymbol{a}\| = (\boldsymbol{a}\cdot\boldsymbol{a})^{\frac{1}{2}}$ and $\|\boldsymbol{A}\| = (\boldsymbol{A} : \boldsymbol{A})^{\frac{1}{2}}$.
Due to the transverse symmetry of a plate, the components of tensor $\boldsymbol{A}$ are constant in this direction thus they are treated as fields referred to Ω. Spectral decomposition of $\boldsymbol{A}$ admits the following form

$$\boldsymbol{A} = \lambda_1\,\boldsymbol{P}_{(1)} + \lambda_2\,\boldsymbol{P}_{(2)} + \lambda_3\,\boldsymbol{P}_{(3)} \tag{7}$$

where $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3$ stand for the Kelvin moduli, and

$$\boldsymbol{P}_{(i)} = \boldsymbol{\omega}_i \otimes \boldsymbol{\omega}_i, \qquad i = 1, 2, 3 \tag{8}$$

denote projectors on eigenspaces corresponding to λ_i. Tensors $\boldsymbol{\omega}_i$ stand for the eigenstates of $\boldsymbol{A}$ satisfying the orthogonality conditions

$$\boldsymbol{\omega}_i \cdot \boldsymbol{\omega}_j = \delta_{ij}. \tag{9}$$

Assuming that the values of moduli $\lambda_1 > \lambda_2 > \lambda_3 > 0$ are fixed within Ω and the orientation of $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, $\boldsymbol{\omega}_3$ is unknown, consider the following optimum design problem: At each point $x \in \Omega$ find such orientation of tensors $\boldsymbol{\omega}_i$ that maximizes the elastic energy density thus minimizing the total compliance of a plate through

$$C_0 = -2 \min_{(\boldsymbol{v},\ v)\in V},\ J_\lambda(\boldsymbol{v},\ v) \tag{10}$$

where $(\boldsymbol{v},\ v) \in V$ denotes the kinematical admissibility of the corresponding test displacement fields and

$$J_\lambda(\boldsymbol{v},\ v) = \int_\Omega W_{\lambda(x)}\left(\boldsymbol{\varepsilon}(\boldsymbol{v}),\ \boldsymbol{\varkappa}(v)\right) dx - f(\boldsymbol{v},\ v) \tag{11}$$

with

$$W_{\lambda(x)}(\boldsymbol{\varepsilon},\ \boldsymbol{\varkappa}) = \frac{1}{2} \max_{\boldsymbol{A}\in\mathcal{T}_{\lambda(x)}} \left\{ \boldsymbol{\varepsilon}\cdot(\boldsymbol{A}\,\boldsymbol{\varepsilon}) + \frac{h^2}{12}\boldsymbol{\varkappa}\cdot(\boldsymbol{A}\boldsymbol{\varkappa}) \right\} \tag{12}$$

where $\mathcal{T}_\lambda(\Omega)$ stands for the set of the constitutive tensor fields determined at each point $x \in \Omega$ by Kelvin moduli $\lambda_1(x)$, $\lambda_2(x)$, $\lambda_3(x)$ and arbitrary eigenstates $\boldsymbol{\omega}_1(x)$, $\boldsymbol{\omega}_2(x)$, $\boldsymbol{\omega}_3(x)$.
Condition of stationarity imposed on the functional in Eq. (11) leads to the equilibrium problem of an effective plate with hyperelastic constitutive properties. Stress and couple resultants are thus calculated as

$$\boldsymbol{N} = \frac{\partial W_{\lambda(x)}(\boldsymbol{\varepsilon},\ \boldsymbol{\varkappa})}{\partial \boldsymbol{\varepsilon}}, \qquad \boldsymbol{M} = \frac{\partial W_{\lambda(x)}(\boldsymbol{\varepsilon},\ \boldsymbol{\varkappa})}{\partial \boldsymbol{\varkappa}}. \tag{13}$$

3. Outline of the optimal energy functional derivation

To make the units of corresponding tensors uniform set

$$\boldsymbol{\kappa} = \frac{h}{\sqrt{12}}\boldsymbol{\varkappa}, \qquad \boldsymbol{K} = \frac{\sqrt{12}}{h}\boldsymbol{M}, \tag{14}$$

which allows for rewriting Eq. (12) in the equivalent form

$$W_\lambda(\boldsymbol{\varepsilon},\ \boldsymbol{\varkappa}) = W_\lambda\left(\boldsymbol{\varepsilon},\ \frac{\sqrt{12}}{h}\boldsymbol{\kappa}\right) = U_\lambda(\boldsymbol{\varepsilon},\ \boldsymbol{\kappa}) \tag{15}$$

where

$$U_\lambda(\boldsymbol{\varepsilon},\ \kappa) = \frac{1}{2}\max_{\boldsymbol{A}\in\mathcal{T}_\lambda} \left\{ \boldsymbol{\varepsilon}\cdot(\boldsymbol{A}\,\boldsymbol{\varepsilon}) + \boldsymbol{\kappa}\cdot(\boldsymbol{A}\,\boldsymbol{\kappa}) \right\} \tag{16}$$

with the argument x being ommited since further discussion is pointwise in Ω.
Formula in Eq. (16) is equivalent to

$$2U(\boldsymbol{\varepsilon},\ \boldsymbol{\kappa}) = \max\left\{ \sum_{k=1}^{3} \lambda_k \left[(\boldsymbol{\omega}_k\cdot\boldsymbol{\varepsilon})^2 + (\boldsymbol{\omega}_k\cdot\boldsymbol{\kappa})^2 \right] \,\middle|\, \boldsymbol{\omega}_i \in \mathbb{E}^2_s,\ \boldsymbol{\omega}_i\cdot\boldsymbol{\omega}_j = \delta_{ij} \right\} \tag{17}$$

and rearranging (17) gives

$$2U_\lambda(\boldsymbol{\varepsilon},\ \boldsymbol{\kappa}) = \lambda_3\left(\|\boldsymbol{\varepsilon}\|^2 + \|\boldsymbol{\kappa}\|^2\right) + 2U_1(\boldsymbol{\varepsilon},\ \boldsymbol{\kappa}) \tag{18}$$

with

$$2U_1(\boldsymbol{\varepsilon},\ \boldsymbol{\kappa}) = \max\left\{ \sum_{\alpha=1}^{2} \mu_\alpha \left[(\boldsymbol{\omega}_\alpha\cdot\boldsymbol{\varepsilon})^2 + (\boldsymbol{\omega}_\alpha\cdot\boldsymbol{\kappa})^2 \right] \,\middle|\, \boldsymbol{\omega}_\alpha \in \mathbb{E}^2_s,\ \boldsymbol{\omega}_\alpha\cdot\boldsymbol{\omega}_\beta = \delta_{\alpha\beta} \right\} \tag{19}$$

where

$$\mu_1 = \lambda_1 - \lambda_3, \qquad \mu_2 = \lambda_2 - \lambda_3. \tag{20}$$

By introducing

$$\xi = \frac{\|\boldsymbol{\kappa}\|^2}{\|\boldsymbol{\varepsilon}\|^2}, \qquad d = \frac{\mu_2}{\mu_1}, \tag{21}$$

and

$$\hat{\boldsymbol{\varepsilon}} = \frac{\boldsymbol{\varepsilon}}{\|\boldsymbol{\varepsilon}\|}, \qquad \hat{\boldsymbol{\kappa}} = \frac{\boldsymbol{\kappa}}{\|\boldsymbol{\kappa}\|} \tag{22}$$

followed by

$$\tilde{\boldsymbol{\kappa}} = \begin{cases} \hat{\boldsymbol{\kappa}} & \text{if} \quad \hat{\boldsymbol{\varepsilon}} \cdot \hat{\boldsymbol{\kappa}} > 0 \\ -\hat{\boldsymbol{\kappa}} & \text{if} \quad \hat{\boldsymbol{\varepsilon}} \cdot \hat{\boldsymbol{\kappa}} < 0 \end{cases} \tag{23}$$

one obtains

$$2U_1(\boldsymbol{\varepsilon},\ \boldsymbol{\kappa}) = \mu_1 \|\boldsymbol{\varepsilon}\|^2 \max \left\{ U(\hat{\boldsymbol{\varepsilon}},\ \tilde{\boldsymbol{\kappa}}) \;\middle|\; \boldsymbol{\omega}_i \in \mathbb{E}_s^2,\ \boldsymbol{\omega}_i \cdot \boldsymbol{\omega}_j = \delta_{ij} \right\} \tag{24}$$

where

$$U(\hat{\boldsymbol{\varepsilon}},\ \tilde{\boldsymbol{\kappa}}) = (\boldsymbol{\omega}_1 \cdot \hat{\boldsymbol{\varepsilon}})^2 + d(\boldsymbol{\omega}_2 \cdot \hat{\boldsymbol{\varepsilon}})^2 + \xi\,(\boldsymbol{\omega}_1 \cdot \tilde{\boldsymbol{\kappa}})^2 + d\,\xi\,(\boldsymbol{\omega}_2 \cdot \tilde{\boldsymbol{\kappa}})^2. \tag{25}$$

Calculations leading to the maximum value of the potential in Eq. (24) are ommited here and will be published elsewhere in full detail. Roughly speaking, the maximum of $U(\hat{\boldsymbol{\varepsilon}},\ \tilde{\boldsymbol{\kappa}})$ is reached if the plane Π_{12}, spanned by vectors $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ is inclined at the angle $\beta = 0$ or $\beta = \pi$ to the plane $\Pi_{\varepsilon\kappa}$ spanned by vectors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\kappa}$. Further analysis is thus performed with an assumption of vectors $\boldsymbol{\varepsilon}$, $\tilde{\boldsymbol{\kappa}}$, $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ being co-planar.
Setting $\alpha = \sphericalangle(\boldsymbol{\varepsilon}, \tilde{\boldsymbol{\kappa}})$, $y_0 = 2\,\alpha$ and, by abuse of notation, $x = \sphericalangle(\boldsymbol{\varepsilon}, \boldsymbol{\omega}_1)$, $y = 2\,x$, see Fig. 1, one may rewrite Eq. (25) as

$$U(\hat{\boldsymbol{\varepsilon}},\ \tilde{\boldsymbol{\kappa}}) = \frac{1}{2}\left[(1+\xi)(1+d) + (1-d) f(y)\right] \tag{26}$$

where

$$f(y) = \cos y + \xi \cos(y - y_0). \tag{27}$$

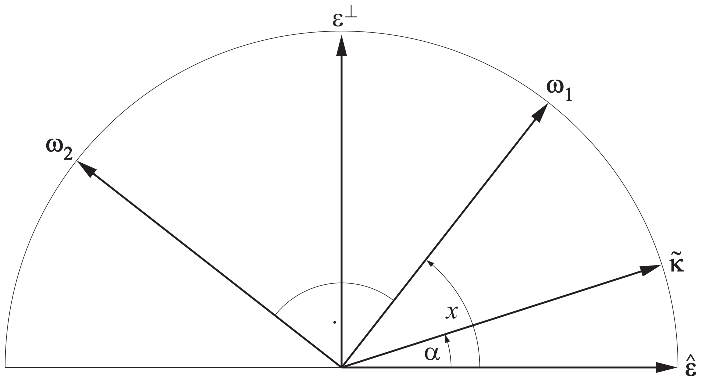

Figure 1: Juxtaposition of vectors $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, $\hat{\boldsymbol{\varepsilon}}$, $\tilde{\boldsymbol{\kappa}}$.

The maximum problem in Eq. (24) is thus reduced to the calculation of $\max_y f(y)$ and it can be shown explicitly that the solution takes the form

$$\max_y f(y) = \sqrt{1 + 2\xi \cos y_0 + \xi^2} \tag{28}$$

with the maximizer expressed by

$$\tan(2x_0) = \frac{\xi \sin(2\alpha)}{1 + \xi \cos(2\alpha)}. \tag{29}$$

Final expressions for energy potentials of an optimal plate are given by

$$2U_1(\boldsymbol{\varepsilon},\ \boldsymbol{\kappa}) = \frac{1}{2}(\mu_1 + \mu_2)(||\boldsymbol{\varepsilon}||^2 + ||\boldsymbol{\kappa}||^2) + \frac{1}{2}(\mu_1 - \mu_2)[(||\boldsymbol{\varepsilon}||^2 - ||\boldsymbol{\kappa}||^2)^2 + 4(\boldsymbol{\varepsilon} \cdot \boldsymbol{\kappa})^2]^{\frac{1}{2}} \tag{30}$$

and

$$U_\lambda(\boldsymbol{\varepsilon},\ \boldsymbol{\kappa}) = \frac{1}{4}(\lambda_1 + \lambda_2)(||\boldsymbol{\varepsilon}||^2 + ||\boldsymbol{\kappa}||^2) + \frac{1}{4}(\lambda_1 - \lambda_2)[(||\boldsymbol{\varepsilon}||^2 - ||\boldsymbol{\kappa}||^2)^2 + 4(\boldsymbol{\varepsilon} \cdot \boldsymbol{\kappa})^2]^{\frac{1}{2}} \tag{31}$$

thus determining $W_\lambda(\boldsymbol{\varepsilon},\ \boldsymbol{\varkappa})$ by Eq. (15) or explicitly

$$W_\lambda(\boldsymbol{\varepsilon},\ \boldsymbol{\varkappa}) = \frac{1}{4}(\lambda_1 + \lambda_2)\left(||\boldsymbol{\varepsilon}||^2 + \frac{h^2}{12}||\boldsymbol{\varkappa}||^2\right) + \frac{1}{4}(\lambda_1 - \lambda_2)\left[\left(||\boldsymbol{\varepsilon}||^2 - \frac{h^2}{12}||\boldsymbol{\varkappa}||^2\right)^2 + \frac{h^2}{3}\left(\boldsymbol{\varepsilon} \cdot \varkappa\right)^2\right]^{\frac{1}{2}}. \tag{32}$$

Properties of thus obtained potential $W_\lambda(\boldsymbol{\varepsilon},\ \boldsymbol{\varkappa})$, including its strict convexity, will be analyzed in separate publication.

4. Constitutive equations of optimal plate

By making use of Eqs (15) and (31) rewrite Eq. (13) in a form

$$\begin{aligned}\boldsymbol{N} &= \frac{1}{2}(\lambda_1+\lambda_2)\left[(1+\nu\,\phi)\,\boldsymbol{\varepsilon}+\nu\,\psi\,\boldsymbol{\kappa}\right],\\ \boldsymbol{K} &= \frac{1}{2}(\lambda_1+\lambda_2)\left[\nu\,\psi\,\boldsymbol{\varepsilon}+(1-\nu\,\phi)\,\boldsymbol{\kappa}\right],\end{aligned} \tag{33}$$

where

$$\nu=\frac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2}, \tag{34}$$

and obviously $0<\nu<1$ as $\lambda_1>\lambda_2>0$.

Next, assume that vectors $\boldsymbol{\varepsilon},\ \boldsymbol{\kappa}$ are not co-linear and set a basis as in Eq. (A.1), see Appendix, and rephrase Eqs (33) thus obtaining

$$\begin{aligned}\boldsymbol{N} &= (A^i{}_j\,\boldsymbol{e}_i\otimes\boldsymbol{e}^j)\,\boldsymbol{\varepsilon}=(A^i{}_j\,\boldsymbol{e}_i\otimes\boldsymbol{e}^j)\,\boldsymbol{e}_1=A^i{}_1\,\boldsymbol{e}_i\\ \boldsymbol{K} &= (A^i{}_j\,\boldsymbol{e}_i\otimes\boldsymbol{e}^j)\,\boldsymbol{\kappa}=(A^i{}_j\,\boldsymbol{e}_i\otimes\boldsymbol{e}^j)\,\boldsymbol{e}_2=A^i{}_2\,\boldsymbol{e}_i.\end{aligned} \tag{35}$$

The comparison of Eqs (33) and (35) leads to the mixed representation of tensor $\boldsymbol{A}$

$$(A^i{}_j)=\begin{bmatrix}\frac{1}{2}(\lambda_1+\lambda_2)(1+\nu\,\phi) & \frac{1}{2}(\lambda_1+\lambda_2)\nu\,\psi & 0\\ \frac{1}{2}(\lambda_1+\lambda_2)\nu\,\psi & \frac{1}{2}(\lambda_1+\lambda_2)(1-\nu\,\phi) & 0\\ 0 & 0 & \lambda_3\end{bmatrix} \tag{36}$$

which means that contravariant representations of $\boldsymbol{N}$ and $\boldsymbol{K}$ in Eqs (33) are given by vectors

$$\begin{bmatrix}N^1\\N^2\\N^3\end{bmatrix}=\frac{1}{2}(\lambda_1+\lambda_2)\begin{bmatrix}1+\nu\,\phi\\ \nu\,\psi\\ 0\end{bmatrix},\quad \begin{bmatrix}K^1\\K^2\\K^3\end{bmatrix}=\frac{1}{2}(\lambda_1+\lambda_2)\begin{bmatrix}\nu\,\psi\\ 1-\nu\,\phi\\ 0\end{bmatrix}. \tag{37}$$

Eigentensors $\boldsymbol{P}_{(i)}$, i.e. proper states of the optimal constitutive tensor $\boldsymbol{A}$, see Eq. (7), can be calculated as, see [7],

$$\begin{aligned}\boldsymbol{P}_{(1)} &= \frac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}(\boldsymbol{A}-\lambda_2\boldsymbol{E})(\boldsymbol{A}-\lambda_3\boldsymbol{E}),\\ \boldsymbol{P}_{(2)} &= \frac{1}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}(\boldsymbol{A}-\lambda_1\boldsymbol{E})(\boldsymbol{A}-\lambda_3\boldsymbol{E}),\\ \boldsymbol{P}_{(3)} &= \frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}(\boldsymbol{A}-\lambda_1\boldsymbol{E})(\boldsymbol{A}-\lambda_2\boldsymbol{E})\end{aligned} \tag{38}$$

or explicitly in the basis $\boldsymbol{e}_j\otimes\boldsymbol{e}^k$

$$(P_{(1)}{}^j{}_k)=\begin{bmatrix}\frac{1}{2}(1+\phi) & \frac{1}{2}\psi & 0\\ \frac{1}{2}\psi & \frac{1}{2}(1-\phi) & 0\\ 0 & 0 & 0\end{bmatrix}, \tag{39}$$

$$(P_{(2)}{}^j{}_k)=\begin{bmatrix}\frac{1}{2}(1-\phi) & -\frac{1}{2}\psi & 0\\ -\frac{1}{2}\psi & \frac{1}{2}(1+\phi) & 0\\ 0 & 0 & 0\end{bmatrix}, \tag{40}$$

and

$$(P_{(3)}{}^{j}{}_{k}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{41}$$

It is a simple matter to check that $\|\boldsymbol{P}_{(i)}\| = 1,\ i = 1, 2, 3.$
Applying Eqs (39), (40) and (A.1), the following formulae

$$\begin{aligned} \psi\|\boldsymbol{\varepsilon}\|^2 + (1-\phi)(\boldsymbol{\varepsilon}\cdot\boldsymbol{\kappa}) = \quad & \psi\|\boldsymbol{\kappa}\|^2 + (1+\phi)(\boldsymbol{\varepsilon}\cdot\boldsymbol{\kappa}) = (\boldsymbol{\varepsilon}\cdot\boldsymbol{\kappa}) + \frac{1}{2}\psi(\|\boldsymbol{\varepsilon}\|^2 + \|\boldsymbol{\kappa}\|^2), \\ -\psi\|\boldsymbol{\varepsilon}\|^2 + (1+\phi)(\boldsymbol{\varepsilon}\cdot\boldsymbol{\kappa}) = & -\psi\|\boldsymbol{\kappa}\|^2 + (1-\phi)(\boldsymbol{\varepsilon}\cdot\boldsymbol{\kappa}) = (\boldsymbol{\varepsilon}\cdot\boldsymbol{\kappa}) - \frac{1}{2}\psi(\|\boldsymbol{\varepsilon}\|^2 + \|\boldsymbol{\kappa}\|^2), \end{aligned} \tag{42}$$

and the expression

$$\cos^2 x_0 = \frac{\boldsymbol{P}_{(1)} : (\boldsymbol{\varepsilon}\otimes\boldsymbol{\varepsilon})}{\|\boldsymbol{\varepsilon}\|^2} \tag{43}$$

allows finding the counterpart of Eq. (29). In this way one obtains

$$\cos^2 x_0 = \frac{1}{2}(1+\phi) + \frac{1}{2}\psi\frac{(\boldsymbol{\varepsilon}\cdot\boldsymbol{\kappa})}{\|\boldsymbol{\varepsilon}\|^2}. \tag{44}$$

Combining the following identity

$$\tan^2(2x_0) = \frac{4(1-\cos^2 x_0)\cos^2 x_0}{2\cos^2 x_0 - 1} \tag{45}$$

with Eq. (44) gives Eq. (29). Similarly, one can calculate

$$\cos^2(x_0 - \alpha) = \frac{\boldsymbol{P}_{(1)} : (\boldsymbol{\kappa}\otimes\boldsymbol{\kappa})}{\|\boldsymbol{\kappa}\|^2} \tag{46}$$

or explicitly

$$\cos^2(x_0 - \alpha) = \frac{1}{2}(1-\phi) + \frac{1}{2}\psi\frac{(\boldsymbol{\varepsilon}\cdot\boldsymbol{\kappa})}{\|\boldsymbol{\kappa}\|^2}, \tag{47}$$

hence the requirement imposed on $\sphericalangle(\hat{\varepsilon},\ \tilde{\boldsymbol{\kappa}})$ being acute can be dropped in the analysis involving $\cos^2 x_0$ and $\cos^2(x_0 - \alpha)$, as the expressions in Eqs (46) and (47) are not sensitive to the sign of $\boldsymbol{\kappa}$.

5. Incremental form of the constitutive equations

Many numerical applications require defining the constitutive equations in an incremental form

$$\begin{aligned} \Delta\boldsymbol{N} &= \frac{\partial\boldsymbol{N}}{\partial\boldsymbol{\varepsilon}}\Delta\boldsymbol{\varepsilon} + \frac{\partial\boldsymbol{N}}{\partial\boldsymbol{\kappa}}\Delta\boldsymbol{\kappa}, \\ \Delta\boldsymbol{K} &= \frac{\partial\boldsymbol{K}}{\partial\boldsymbol{\varepsilon}}\Delta\boldsymbol{\varepsilon} + \frac{\partial\boldsymbol{K}}{\partial\boldsymbol{\kappa}}\Delta\boldsymbol{\kappa}. \end{aligned} \tag{48}$$

Easy computations involving formulae in Eqs (33) show that derivatives in Eqs (48) can be expressed in a form

$$\begin{aligned} \frac{\partial\boldsymbol{N}}{\partial\boldsymbol{\varepsilon}} &= \frac{1}{2}(\lambda_1+\lambda_2)\left[\nu\,\frac{\partial\phi}{\partial\boldsymbol{\varepsilon}}\otimes\boldsymbol{\varepsilon} + \nu\,\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}}\otimes\boldsymbol{\kappa} + (1+\nu\,\phi)\,\boldsymbol{I}\right], \\ \frac{\partial\boldsymbol{N}}{\partial\boldsymbol{\kappa}} &= \frac{1}{2}(\lambda_1+\lambda_2)\left[\nu\,\frac{\partial\phi}{\partial\boldsymbol{\kappa}}\otimes\boldsymbol{\varepsilon} + \nu\,\frac{\partial\psi}{\partial\boldsymbol{\kappa}}\otimes\boldsymbol{\kappa} + \nu\,\psi\,\boldsymbol{I}\right], \\ \frac{\partial\boldsymbol{K}}{\partial\boldsymbol{\varepsilon}} &= \frac{1}{2}(\lambda_1+\lambda_2)\left[\nu\,\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}}\otimes\boldsymbol{\varepsilon} - \nu\,\frac{\partial\phi}{\partial\boldsymbol{\varepsilon}}\otimes\boldsymbol{\kappa} + \nu\,\psi\,\boldsymbol{I}\right], \\ \frac{\partial\boldsymbol{K}}{\partial\boldsymbol{\kappa}} &= \frac{1}{2}(\lambda_1+\lambda_2)\left[\nu\,\frac{\partial\psi}{\partial\boldsymbol{\kappa}}\otimes\boldsymbol{\varepsilon} - \nu\,\frac{\partial\phi}{\partial\boldsymbol{\kappa}}\otimes\boldsymbol{\kappa} + (1-\nu\,\phi)\,\boldsymbol{I}\right], \end{aligned} \tag{49}$$

where $\boldsymbol{I}$ stands for the unit tensor, see Eq. (A.7), and

$$
\begin{aligned}
\frac{\partial \phi}{\partial \boldsymbol{\varepsilon}} &= 2\frac{\psi}{G}\left(\psi\,\boldsymbol{\varepsilon} - \phi\,\boldsymbol{\kappa}\right), & \frac{\partial \phi}{\partial \boldsymbol{\kappa}} &= -2\frac{\psi}{G}\left(\phi\,\boldsymbol{\varepsilon} + \psi\,\boldsymbol{\kappa}\right), \\
\frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} &= -2\frac{\phi}{G}\left(\psi\,\boldsymbol{\varepsilon} - \phi\,\boldsymbol{\kappa}\right), & \frac{\partial \psi}{\partial \boldsymbol{\kappa}} &= 2\frac{\phi}{G}\left(\phi\,\boldsymbol{\varepsilon} + \psi\,\boldsymbol{\kappa}\right).
\end{aligned} \tag{50}
$$

In the calculations one also needs to take into account the representation of the dyadic products $\boldsymbol{\varepsilon}\otimes\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}\otimes\boldsymbol{\kappa}$, $\boldsymbol{\kappa}\otimes\boldsymbol{\kappa}$ and tensor $\boldsymbol{I}$ in a basis chosen for computations. For this purpose assume that $\{\boldsymbol{i}_1,\ \boldsymbol{i}_2\}$ constitute a fixed orthonormal basis in $\mathbb{R}^2$ and note that $\{\boldsymbol{i}_1\otimes\boldsymbol{i}_1\,,\ \boldsymbol{i}_2\otimes\boldsymbol{i}_2,\ (1/\sqrt{2})(\boldsymbol{i}_1\otimes\boldsymbol{i}_2+\boldsymbol{i}_2\otimes\boldsymbol{i}_1)\}$ form an orthonormal basis in $\mathbb{R}^3$ which is convenient for describing two-dimensional symmetric tensors as three-dimensional vectors, see Eqs (5), thus obtaining

$$
\boldsymbol{\varepsilon}\otimes\boldsymbol{\varepsilon} = \begin{bmatrix} (\varepsilon_{11})^2 & \varepsilon_{11}\varepsilon_{22} & \sqrt{2}\varepsilon_{11}\varepsilon_{12} \\ \varepsilon_{22}\varepsilon_{11} & (\varepsilon_{22})^2 & \sqrt{2}\varepsilon_{22}\varepsilon_{12} \\ \sqrt{2}\varepsilon_{12}\varepsilon_{11} & \sqrt{2}\varepsilon_{12}\varepsilon_{22} & 2\,(\varepsilon_{12})^2 \end{bmatrix}, \tag{51}
$$

$$
\boldsymbol{\varepsilon}\otimes\boldsymbol{\kappa} = \begin{bmatrix} \varepsilon_{11}\kappa_{11} & \varepsilon_{11}\kappa_{22} & \sqrt{2}\varepsilon_{11}\kappa_{12} \\ \varepsilon_{22}\kappa_{11} & \varepsilon_{22}\kappa_{22} & \sqrt{2}\varepsilon_{22}\kappa_{12} \\ \sqrt{2}\varepsilon_{12}\kappa_{11} & \sqrt{2}\varepsilon_{12}\kappa_{22} & 2\,\varepsilon_{12}\kappa_{12} \end{bmatrix}, \tag{52}
$$

$$
\boldsymbol{\kappa}\otimes\boldsymbol{\kappa} = \begin{bmatrix} (\kappa_{11})^2 & \kappa_{11}\kappa_{22} & \sqrt{2}\kappa_{11}\kappa_{12} \\ \kappa_{22}\kappa_{11} & (\kappa_{22})^2 & \sqrt{2}\kappa_{22}\kappa_{12} \\ \sqrt{2}\kappa_{12}\kappa_{11} & \sqrt{2}\kappa_{12}\kappa_{22} & 2\,(\kappa_{12})^2 \end{bmatrix}, \tag{53}
$$

with $(\boldsymbol{\varepsilon}\otimes\boldsymbol{\kappa})_{\alpha\beta} = (\boldsymbol{\kappa}\otimes\boldsymbol{\varepsilon})_{\beta\alpha}$ and

$$
\boldsymbol{\omega}_3\otimes\boldsymbol{\omega}_3 = \begin{bmatrix} (q_1)^2 & q_1\,q_2 & q_1\,q_3 \\ q_2\,q_1 & (q_2)^2 & q_2\,q_3 \\ q_3\,q_1 & q_3\,q_2 & (q_3)^2 \end{bmatrix}, \tag{54}
$$

where

$$
\begin{aligned}
q_1 &= \frac{\varepsilon_2\kappa_3 - \varepsilon_3\kappa_2}{\sqrt{(\varepsilon_1\kappa_2-\varepsilon_2\kappa_1)^2 + (\varepsilon_2\kappa_3-\varepsilon_3\kappa_2)^2 + (\varepsilon_1\kappa_3-\varepsilon_3\kappa_1)^2}}, \\
q_2 &= -\frac{\varepsilon_1\kappa_3 - \varepsilon_3\kappa_1}{\sqrt{(\varepsilon_1\kappa_2-\varepsilon_2\kappa_1)^2 + (\varepsilon_2\kappa_3-\varepsilon_3\kappa_2)^2 + (\varepsilon_1\kappa_3-\varepsilon_3\kappa_1)^2}}, \\
q_3 &= \frac{\varepsilon_1\kappa_2 - \varepsilon_2\kappa_1}{\sqrt{(\varepsilon_1\kappa_2-\varepsilon_2\kappa_1)^2 + (\varepsilon_2\kappa_3-\varepsilon_3\kappa_2)^2 + (\varepsilon_1\kappa_3-\varepsilon_3\kappa_1)^2}},
\end{aligned} \tag{55}
$$

and

$$
\begin{aligned}
\varepsilon_1 &= \varepsilon_{11}, & \kappa_1 &= \kappa_{11}, \\
\varepsilon_2 &= \varepsilon_{22}, & \kappa_2 &= \kappa_{22}, \\
\varepsilon_3 &= \sqrt{2}\varepsilon_{12}, & \kappa_3 &= \sqrt{2}\kappa_{12}.
\end{aligned} \tag{56}
$$

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Appendix

Introduce a basis

$$\begin{aligned} \boldsymbol{e}_1 &= \boldsymbol{\varepsilon} \\ \boldsymbol{e}_2 &= \boldsymbol{\kappa} \\ \boldsymbol{e}_3 &= \frac{\boldsymbol{\varepsilon} \times \boldsymbol{\kappa}}{\|\boldsymbol{\varepsilon} \times \boldsymbol{\kappa}\|}, \end{aligned} \tag{A.1}$$

such that

$$\begin{aligned} \boldsymbol{e}_1 \cdot \boldsymbol{e}_3 &= 0, \\ \boldsymbol{e}_2 \cdot \boldsymbol{e}_3 &= 0, \\ \|\boldsymbol{e}_3\| &= 1. \end{aligned} \tag{A.2}$$

Next, calculate the covariant components $E_{ij} = E_{ji}$ of a metric tensor $\boldsymbol{E} = E_{ij}\boldsymbol{e}^i \otimes \boldsymbol{e}^j$

$$\begin{array}{ll} E_{11} = \|\boldsymbol{\varepsilon}\|^2, & E_{13} = E_{23} = 0, \\ E_{12} = \boldsymbol{\varepsilon} \cdot \boldsymbol{\kappa}, & E_{33} = 1, \\ E_{22} = \|\boldsymbol{\kappa}\|^2, & \end{array} \tag{A.3}$$

and recall that mixed components of $\boldsymbol{E}$ are given by formula $E^i{}_j = E^j{}_i = \delta^i{}_j$, where

$$\delta^i{}_j = \boldsymbol{e}_i \cdot \boldsymbol{e}^j = E_{ik}\boldsymbol{e}^k \cdot \boldsymbol{e}^j = E_{ik}E^{kj}. \tag{A.4}$$

Making use of Eqs (A.3) and (A.4) allows for the calculation of contravariant components $E^{ij} = E^{ji}$

$$\begin{array}{ll} E^{11} = \dfrac{\|\boldsymbol{\kappa}\|^2}{\|\boldsymbol{\varepsilon}\|^2\,\|\boldsymbol{\kappa}\|^2 - (\boldsymbol{\varepsilon} \cdot \boldsymbol{\kappa})^2}, & E^{13} = E^{23} = 0, \\ E^{12} = -\dfrac{\boldsymbol{\varepsilon} \cdot \boldsymbol{\kappa}}{\|\boldsymbol{\varepsilon}\|^2\,\|\boldsymbol{\kappa}\|^2 - (\boldsymbol{\varepsilon} \cdot \boldsymbol{\kappa})^2}, & E^{33} = 1, \\ E^{22} = \dfrac{\|\boldsymbol{\varepsilon}\|^2}{\|\boldsymbol{\varepsilon}\|^2\,\|\boldsymbol{\kappa}\|^2 - (\boldsymbol{\varepsilon} \cdot \boldsymbol{\kappa})^2}, & \end{array} \tag{A.5}$$

and co-basis vectors $\boldsymbol{e}^i = E^{ij}\,\boldsymbol{e}_j$

$$\begin{aligned} \boldsymbol{e}^1 &= E^{11}\,\boldsymbol{e}_1 + E^{12}\,\boldsymbol{e}_2 = \frac{\|\boldsymbol{\kappa}\|^2}{\|\boldsymbol{\varepsilon}\|^2\,\|\boldsymbol{\kappa}\|^2 - (\boldsymbol{\varepsilon} \cdot \boldsymbol{\kappa})^2}\,\boldsymbol{\varepsilon} - \frac{\boldsymbol{\varepsilon} \cdot \boldsymbol{\kappa}}{\|\boldsymbol{\varepsilon}\|^2\,\|\boldsymbol{\kappa}\|^2 - (\boldsymbol{\varepsilon} \cdot \boldsymbol{\kappa})^2}\,\boldsymbol{\kappa}, \\ \boldsymbol{e}^2 &= E^{21}\,\boldsymbol{e}_1 + E^{22}\,\boldsymbol{e}_2 = -\frac{\boldsymbol{\varepsilon} \cdot \boldsymbol{\kappa}}{\|\boldsymbol{\varepsilon}\|^2\,\|\boldsymbol{\kappa}\|^2 - (\boldsymbol{\varepsilon} \cdot \boldsymbol{\kappa})^2}\,\boldsymbol{\varepsilon} + \frac{\|\boldsymbol{\varepsilon}\|^2}{\|\boldsymbol{\varepsilon}\|^2\,\|\boldsymbol{\kappa}\|^2 - (\boldsymbol{\varepsilon} \cdot \boldsymbol{\kappa})^2}\,\boldsymbol{\kappa}, \\ \boldsymbol{e}^3 &= E^{33}\,\boldsymbol{e}_3 = \frac{\boldsymbol{\varepsilon} \times \boldsymbol{\kappa}}{\|\boldsymbol{\varepsilon} \times \boldsymbol{\kappa}\|}. \end{aligned} \tag{A.6}$$

In this notation, the unit Hooke tensor $\boldsymbol{I}$ is expressed by

$$\boldsymbol{I} = \boldsymbol{e}_i \otimes \boldsymbol{e}^i \tag{A.7}$$

i.e. coincides with the metric tensor $\boldsymbol{E}$ in mixed representation.